Efficient Investment and Search in Matching Markets*

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Abstract

We study a model where heterogeneous agents invest in skills and then enter a two-sided matching market which has search frictions. In every period, agents incur an additive search cost, pairs meet at random, and can either accept and bargain over the joint output or reject and continue searching for a better match. Potential sources for inefficiencies are the hold-up problem and mismatches between skills. Despite these, we prove a second welfare theorem: the *constrained efficient* allocation is an equilibrium. Furthermore, we establish a general assortative matching result, provide conditions for equilibrium uniqueness, and characterize equilibria in symmetric two-skill economies. Finally, we show that the efficient outcome can be discriminatory in the marriage market.

1 Introduction

This paper studies two-sided matching markets whose participants engage in costly search for partners and make investments before entry. Our goal is to understand how search frictions impact the equilibrium allocation, particularly the investment decisions, the matching outcome, and social welfare.

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Pre-match investment and search are important features of various market settings. For example, individuals in the marriage market make premarital investments in their education and career before looking for partners. In the labor market, workers acquire human capital before searching for jobs, while firms adopt technologies before hiring workers. A similar situation occurs in other settings: in the real estate market, developers often build before finding prospective buyers; in a financial market, entrepreneurs invest time and money developing start-ups prior to seeking venture capital funding; and in product markets, sellers make investments in quality before seeking potential buyers.

In such settings, the agents are usually heterogeneous and the payoff from an investment depends both on who matches with whom and on the search duration. For example, in the labor market, a worker's return to schooling depends on which types of firms may potentially hire her and how long it will take her to find a job. Likewise, a firm's benefit from adopting a new technology depends on the skills of workers it may potentially hire. Therefore, search frictions not only impact the matching outcome but also the incentives to invest.

Previous work has extensively analyzed the incentives to invest when agents enter a perfectly competitive (frictionless) matching market (see, e.g., Cole et al. 2001, Noldeke and Samuelson 2015). We contribute to the literature by introducing costly search and decentralized price determination.

We develop a model of matching with transfers between two populations of agents, called buyers and sellers, but one can equally consider workers and firms, men and women, or any other two populations that invest and then match. What is important is that output is produced by pairs of agents, one from each side of the market. The model has two key ingredients. First, agents invest in skills before entering the market. The investment costs are heterogeneous and buyer-seller pairs produce output according to their skills. Second, the trading process is decentralized in the sense that the agents search for partners and bargain pairwise over prices.

We consider a standard random search and bargaining process with additive search costs rather than discounting, as in Atakan [2006]. Specifically, in every period, each agent incurs the same search cost and randomly meets an agent from the other side of the market. When two agents meet, they can either agree to match or continue searching. If both agree to match, they exit the market and divide the output according to Nash bargaining. A new cohort of agents is born in every period; each new-born agent acquires a skill and then enters the market. We analyze a *steady-state equilibrium* where, for every skill, the inflow of agents to the market equals the outflow, so that the distribution of skills in the market is in a steady state.

The term "skill" refers to investments that enhance productivity. For instance, in the labor market, a worker may invest in acquiring some level of education (the worker's skill), while a firm may invest in adopting a particular technology (the firm's skill). In a product market, a seller may invest in order to reduce their production cost (the seller's skill), a buyer may invest to increase her consumption value (the buyer's skill), and the joint production function is the difference between the buyer's value and the seller's cost. In a marriage market, the joint production function depends on the premarital investments made by both agents and we assume ex-ante symmetry, that is, men and women can acquire the same skills and they have the same cost distribution.

The market is competitive in that every skill has a value and agents optimize given these values. Thus, an agent will acquire a better skill if its marginal value exceeds the agent's marginal cost, and when two agents meet they will accept (reject) the match if their joint output is greater (smaller) than the sum of their values. As in standard search and matching models, these values are endogenously determined in an equilibrium and must be consistent with the steady-state conditions, the search strategies, and Nash bargaining (see e.g., Burdett and Coles 1999; Shimer and Smith 2000). The most important and novel feature of our model is that the values serve *double duty*: creating incentives to invest and to accept matches.

There are three potential sources of inefficiencies. First, since investments are sunk by the time agents meet, a hold-up problem may distort the incentive to invest (because agents bear the entire cost of investment and receive only a fraction of the additional output). Second, agents are heterogeneous and mismatches between skills can occur. For instance, a social planner may want an agent to accept a partner who is not very productive, but the agent would rather search for a more productive match. Third, there may exist multiple equilibria, and the agents fail to coordinate on the efficient one.

This paper has three main points. First, despite potential inefficiencies in both investment and matching, we prove a version of the second welfare theorem: for any search cost, the constrained efficient allocation is an equilibrium outcome.[1] The welfare theorem also establishes the existence of equilibrium. The proof demonstrates that there are market values which perfectly align the incentives of the agents with those of the planner, while also satisfying the standard equilibrium conditions. Regarding matching, notice that two agents accept each other precisely when their output exceeds the sum of their values, whereas a social planner trades off their match output versus the output of matching them with other agents, while also taking into account the impact on the total search cost and the steady-state distribution. The fact that the efficient matching rule can be decentralized is significant because it extends Hosios' (1990) classic efficiency result to a setting with heterogeneous agents.

Turning to investment, agents acquire skills by comparing their marginal cost to the marginal value of skills in the market. Therefore, for efficiency, the slope of the value function must coincide with the slope of the social welfare function. It is striking that these values *simultaneously* solve the investment and matching problems. The intuition for why the hold-up problem is resolved is that matched agents divide their output according to their values from continuing to search. By investing in a better skill, an agent enhances the productivity of every potential match and also improves her value, thereby capturing a larger fraction of the additional output.

Our second main point is that the equilibria have a clear and simple structure. We prove that there is assortative matching if the production function is super/submodular. Furthermore, if the production function is additively separable, then the equilibrium is unique and it achieves the first-best allocation. Economies with non-separable production functions can have multiple equilibria and the agents may fail to coordinate on the efficient one.[2] To better understand these coordination issues, we characterize equilibria in a symmetric economy (a marriage market) with two skills.

The third main point is that in this marriage market, when there is strong submodularity and a high search cost, an asymmetric equilibrium not only exists, but is in fact *constrained efficient*. That is, the agents on opposite sides are ex-ante identical, but they receive different payoffs from investing. Conversely, when the production

[1]The constrained efficient allocation solves the problem faced by a social planner who controls the agents' decisions while respecting the steady-state condition. Since utility is transferable, the Pareto-optimal outcomes are the constrained efficient ones.

[2]For example, a no-investment equilibrium may occur if not investing is self-reinforcing: agents do not invest because all others do not.

function is supermodular or the search cost is low, every (interior) equilibrium is symmetric, and hence the efficient allocation is as well.

The efficiency of the asymmetric equilibrium hinges on the fundamental tradeoff between investment costs and search costs: in the asymmetric equilibrium, agents from opposite sides invest differently, which induces a higher total investment cost (due to a misallocation of talent as high-cost buyers invest instead of low-cost sellers, or vice versa), but when the production function is submodular, the lopsided skill distributions facilitate search (since it is more likely that agents with opposite skills meet). The latter outweighs the former if and only if there is strong submodularity and substantial search costs.

These results have practical implications. For example, in the marriage market, we establish when discrimination can occur, which generates a gender-gap in skill-acquisition. In the labor market, we establish when sorting occurs (high-tech firms match with high-skill workers), and the model captures how the degree of sorting affects (and is affected by) investment. For instance, a lower search cost generally leads to finer sorting, which affects the marginal productivity of some skills and thereby the incentive to invest. Finally, in product markets, the joint output function is often assumed additively separable, and we show that there exists a unique equilibrium which achieves the first-best allocation.

Related Literature

Previous work on two-sided matching with transfers has extended the classical assignment model of Shapley and Shubik [1971] to settings with ex-ante investments. There is perfect frictionless matching in these models, and it is typically found that the first-best allocation is a competitive equilibrium outcome, but there may exist additional inefficient equilibria (see, e.g., Cole et al. 2001, Mailath et al. 2013, Noldeke and Samuelson 2015, Dizdar 2018).[3] In contrast, we study a frictional matching market where the first-best allocation is typically infeasible. We contribute to this literature by establishing that the *constrained efficient* allocation is an equilibrium outcome. Furthermore, the tradeoff between productivity, search cost, and investment cost has

[3]Our model has transferrable utility and models of non-transferable utility are less relevant to ours, but it is important to mention that efficiency is not always attained there (see, e.g., Bhaskar and Hopkins 2016).

novel implications. For example, in a symmetric two-skill economy with a (strongly) submodular production function, the constrained efficient equilibrium is asymmetric (discriminatory) when search costs are high, whereas every equilibrium is symmetric when search costs are low (see Propositions 3-5 in Section 5.3)

We contribute to the literature on random search and matching with transferable utility (see, e.g., Burdett and Coles [1999], Shimer and Smith [2000], and Atakan [2006]). First, we extend these models by introducing pre-match investment, and so agents' skills are endogenous. Second, we show that there is an equilibrium in which both investment and matching are constrained efficient. Third, most previous work studies sorting in a single population search market (e.g., Atakan, 2006, Shimer and Smith, 2000) and we extend the existing results by establishing assortative matching in a truly two-sided matching market, such as labor or product markets.[4] Finally, the Welfare Theorem establishes the existence of equilibrium and provides a useful computational tool since the planner's problem is often more amenable to numerical analysis than the equilibrium conditions.[5]

Our paper contributes to the literature on efficiency in decentralized exchange initiated by the classic work of Rubinstein and Wolinsky [1985, 1990]. Gale's [1987] paper and the subsequent work by Mortensen and Wright [2002] and Lauermann [2013] investigate economies with *quasi-linear utility* in which buyers and sellers meet at random and bargain pairwise over the price of a single homogeneous good. They demonstrate that as the discount factor goes to one, the equilibrium converges to the (flow) Walrasian outcome. Moreover, in the limit equilibria, a law of one price prevails: all meetings result in a trade and all trades occur at the same price. We consider a setting with a general output function, ex-ante investment, and matching. In equilibrium, typically, some matches are rejected and there is price dispersion. Nevertheless, the constrained efficient allocation is an equilibrium.

Hosios' [1990] classic paper considers a standard search and bargaining model with *homogeneous* agents who choose their search intensity. He proves that the equilibrium can achieve the constrained efficient outcome, provided that the meeting function

[4]That is, agents on opposite sides of the market may acquire different types of skills, face different investment costs, and incur different search costs. Moreover, even if the two sides of the market are symmetric a priori, the endogenously chosen skill distributions might be asymmetric in equilibrium.

[5]In other search models, the existence of steady-state equilibria can be difficult to establish (see, e.g., Manea 2017 and Lauermann et al. 2020).

exhibits constant returns to scale and the bargaining weight equals the elasticity of the meeting function. The key point is that in the equilibrium with this bargaining weight, the search externalities that agents impose on each other are perfectly offset. We derive an analogous result for an economy with heterogeneous agents and pre-match investment (see Proposition 5). In particular, we demonstrate that the equilibrium values simultaneously resolve both the hold-up problem and the matching problem.

The assumption that agents incur the same additive search cost, rather than discounting time, is crucial for our results. In a search model with discounting, when an agent rejects a match, she incurs a search cost proportional to her continuation value because her payoff is delayed. These search costs are heterogeneous (skill-specific) and as we show in Section 6.4, they create inefficiencies: investments are always inefficient, matching can be inefficient and even nonassortative. In search-and-matching models with discounting, previous work shows that the hold-up problem distorts agents' incentives to invest and enter the market (see Acemoglu 1996, Masters 1998, Acemoglu and Shimer 1999a), and agents do not fully internalize the externalities they impose on each other when they accept and reject partners (see Shimer and Smith 2001). Furthermore, the equilibria may fail to exhibit sorting (see Shimer and Smith 2000, Atakan 2006). The key difference is that the additive search cost model severs the implicit link between search costs and values. Our findings suggest that inefficiencies are not the result of search frictions per se, but rather heterogeneity in search frictions.

It is well known that when sellers can post prices to attract buyers, the equilibrium can overcome both the hold-up and matching problems (see, e.g., Acemoglu and Shimer 2000, 1999b, Shi 2001, Jerez 2017). However, in directed search models, the search and matching process and the price-determination mechanism are substantially different than in random search and bargaining models. We are the first to show that the constrained efficient allocation can be decentralized as an equilibrium for the latter class of models.

Finally, we also contribute to the literature on premarital investment in the marriage market (see, e.g., Peters and Siow 2002, Fernandez et al. 2005, Nick and Walsh 2007, Chiappori et al. 2009). Previous work without investments or search has shown that asymmetric equilibria can arise due to complementarities in the household production function (see, e.g., K.Hadfield 1999). To the best of our knowledge, we are the first to establish the efficiency of an asymmetric equilibrium in a search and matching environment. The model also provides a useful framework to do comparative statics and to analyze the gender gap. In models of statistical discrimination (Norman 2003), and directed search (Mailath et al. 2000), ex-ante identical groups can receive different outcomes in the market, but the mechanisms that deliver these unequal outcomes differ from ours. The current analysis highlights the tradeoff between productivity, investment, and search costs.

2 The Model

There is a continuum population of buyers $\beta \sim F^b$ and sellers $\sigma \sim F^s$. Each buyer chooses one skill from a finite set $I \subset \mathbb{N}$ and each seller chooses one skill from a finite set $J \subset \mathbb{N}$. The cost of skill i to buyer β is $C^b(i, \beta)$ and the cost of skill j to seller σ is $C^s(j, \sigma)$. Output is produced by buyer-seller pairs according to their skills and is summarized by the matrix $G = [g_{ij}]$, where the entry $g_{ij} \geq 0$ denotes the output of a pair with skills i, j. Agents have transferable utility and incur a fixed per-period search cost $c > 0$.

The type distributions F^b and F^s are continuous and strictly increasing over their connected supports: $\mathcal{B} = supp(F^b) \subseteq \mathbb{R}$ and $\mathcal{S} = supp(F^s) \subseteq \mathbb{R}$. The match output g_{ij} is strictly increasing in skills. The cost functions are non-negative, strictly increasing, bounded and continuous. Furthermore, they satisfy increasing differences: the difference $C^b(i', \beta) - C^b(i, \beta)$ is strictly increasing in β whenever $i' > i$ and the difference $C^s(j', \sigma) - C^s(j, \sigma)$ is strictly increasing in σ whenever $j' > j$. That is, a higher skill enhances match output, but is more costly to acquire, and higher types have higher costs and higher marginal costs.

Definition. An *economy* is a tuple $\langle F^b, F^s, I, J, C^b, C^s, G, c\rangle$ consisting of prior distributions, skill sets, investment cost functions, the output function, and a search cost. The economy is *symmetric* if $F^b = F^s, I = J$, $C^b = C^s$, and $g_{ij} = g_{ji}, \forall i, j$.

Timing. Search and matching takes place in discrete time periods over an infinite horizon. In every period, a unit measure of buyers and a unit measure of sellers are born. Each newborn agent chooses a skill and then enters the matching market. Each agent in the market incurs the search cost c and randomly meets a partner. When two agents meet, they can either accept the match or continue searching in the hope of finding a better partner. If both agents accept the match, then they exit the market and divide their output according to Nash bargaining. If at least one rejects, then they both remain in the market. In the next period, a new cohort enters the market and the process repeats itself. We refer to the agents in the market as the stock population, the agents entering the market as the inflow population, and the agents exiting the market as the outflow population.

Steady State. The economy is in a *steady state* if in the stock population the measure of agents with each skill is constant over time. Therefore, for each skill, the inflow of agents equals the outflow. In a steady state, we denote the measures of skill i buyers and skill j sellers in the stock population by b_i and s_j. The total measures of buyers and sellers in the market are $B = \sum_{i \in I} b_i$ and $S = \sum_{j \in J} s_j$, and the *proportions* of skill i buyers and skill j sellers are $x_i = b_i/B$ and $y_j = s_j/S$ (notice that $B \geq 1$ and $S \geq 1$). The notation (x_i) and (y_j) denotes the profile of buyer and seller proportions. We let $z = \langle (x_i), (y_j), B, S \rangle$ be the *state variable* where the set of all state variables is $\mathcal{Z} = \Delta(I) \times \Delta(J) \times [1, \infty)^2$.

Meetings. An agent can meet at most one partner in each period and pairs meet at random. The total number of meetings per period is $\mu(B, S) = \min(B, S)$. Therefore, if the market is balanced, i.e. $B = S$, then every agent randomly draws a partner in each period. For now, we will assume that the market is balanced, and denote the market size by $N = B = S$ and the state by $z = \langle (x_i), (y_j), N \rangle$. If the market is unbalanced, agents on the long side of the market would need to be rationed, but this cannot occur in equilibrium (see Lemma 1). In Section 6, we extend the analysis to consider more general meeting functions.

Strategies. An agent's strategy specifies their choice of skill and which agents they accept. We assume Markov strategies. The *investment strategy* of buyer β is $\mathcal{I}^\beta : \mathcal{Z} \to I$ and that of seller σ is $\mathcal{I}^\sigma : \mathcal{Z} \to J$. The *acceptance strategy* of a buyer with skill i is $A_i^b : \mathcal{Z} \times J \to [0, 1]$, which specifies the probability she accepts a seller with skill j upon meeting. For a seller with skill j, it is $A_j^s : \mathcal{Z} \times I \to [0, 1]$. Note that the acceptance strategies do not depend on the agents' identities because the match output depends only on skills. To simplify, we will suppress the state variable in the strategies. It will be convenient to summarize the acceptance strategies by a matching matrix $M = [m_{ij}]$, where the element $m_{ij} = A_i^b(j) \cdot A_j^s(i)$ is the probability that buyer i and seller j both agree to match, conditional on meeting.

2.1 Equilibrium

Every skill has a value in the market and agents optimize given the values and the steady state. We denote the values of a skill i buyer by v_i, and of a skill j seller by w_j. The profiles of buyer and seller values are (v_i) and (w_j), respectively. As is standard in the search and matching literature, we define an equilibrium using the matching matrix and values, rather than the strategies.

Definition. A *steady state equilibrium* $\langle z, M, (v_i), (w_j) \rangle$ consists of a state variable, a matching matrix, and market values satisfying conditions (1), (3), and (4) below.

The first condition is that acceptance decisions are individually optimal. When two agents with skills i and j meet, the *surplus* is $s_{ij} = g_{ij} - v_i - w_j$, and the acceptance decisions satisfies the *Efficient Matching* condition:

$$m_{ij} = \begin{cases} 1 & \text{if } s_{ij} > 0 \\ 0 & \text{if } s_{ij} < 0 \end{cases} \tag{1}$$

The condition is intuitive because an agent will accept a match precisely when her payoff from doing so is greater than her continuation value. When the surplus is negative, i.e. $v_i + w_j > g_{ij}$, the match is always rejected because both agents cannot receive at least their value, while when the surplus is positive, the agents will reach a mutually beneficial agreement. If the surplus is exactly zero, then m_{ij} is unrestricted, i.e. $0 \leq m_{ij} \leq 1$.

When two agents accept each other, each receives their own value and half of the match surplus. This division rule is the Nash bargaining solution and also is a subgame perfect equilibrium of a strategic bargaining game (see, e.g., Atakan 2006). The second condition is that the values are self-consistent, and therefore satisfy the following recursive equation:

$$\begin{aligned} v_i &= \sum_{j \in J} y_j \left[m_{ij} \left(v_i + \tfrac{s_{ij}}{2} \right) + (1 - m_{ij}) v_i \right] - c, \quad \forall i \\ w_j &= \sum_{i \in I} x_i \left[m_{ij} \left(w_j + \tfrac{s_{ij}}{2} \right) + (1 - m_{ij}) w_j \right] - c, \quad \forall j \end{aligned} \tag{2}$$

That is, in every period, buyer i pays the search cost c and meets seller j with probability y_j. If a match is accepted, the buyer receives her continuation value and half of the surplus, whereas if the match is rejected, she attains her continuation value

v_i. Simplifying, we obtain the *Constant Surplus* equations:

$$\sum_{j\in J} y_j m_{ij} s_{ij} = 2c, \forall i \qquad (3)$$
$$\sum_{i\in I} x_i m_{ij} s_{ij} = 2c, \forall j$$

The investment decisions are individually optimal: $\mathcal{I}^\beta \in \arg\max_{i\in I} v_i - C(i,\beta), \forall\beta$ and $\mathcal{I}^\sigma \in \arg\max_{j\in J} w_j - C(j,\sigma), \forall\sigma$. Since the cost function satisfies strictly increasing differences, the set of cost types who choose each skill is an interval (and hence measurable). Furthermore, at most one type can be indifferent between any two skills,[6] and thus the values (v_i) and (w_j) uniquely determine the inflows (up to measure zero). Formally, we denote by $F^b(A) = \int_A dF^b$ the measure of set A according to F^b. The measure of buyers who choose skill i is $F^b\left(\left\{\beta : \mathcal{I}^\beta = i\right\}\right) = F^b\left(\left\{\beta : i \in \arg\max_{i'\in I} v_{i'} - C^b(i',\beta)\right\}\right)$, and analogously for sellers.

The final set of conditions is that the economy is in a steady state. We refer to Equations (4) as the *Inflow=Outflow* equations:

$$\overbrace{F^b\left(\left\{\beta : i \in \arg\max_{i'\in I} v_{i'} - C^b(i',\beta)\right\}\right)}^{inflow} = \overbrace{Nx_i \sum_{j\in J} y_j m_{ij}}^{outflow}, \forall i \in I \qquad (4)$$
$$F^s\left(\left\{\sigma : j \in \arg\max_{j'\in j} w_{j'} - C^s(j',\sigma)\right\}\right) = Ny_j \sum_{i\in I} x_i m_{ij}, \forall j \in J$$

The inflow is the measure of buyers who choose skill i. The outflow is the measure of skill i buyers in the market, Nx_i, times the probability of exiting (each buyer meets a skill j with probability, y_j, and they accept each other with probability, m_{ij}). The seller Inflow=Outflow equations are analogous.

2.2 Equilibrium Properties

The next two lemmas will be useful. The first states that unbalanced states do not occur in equilibria.

[6] If buyer $\hat\beta$ is indifferent between acquiring skills i and i', where $i' > i$, then all buyers $\beta < \hat\beta$ strictly prefer skill i' to skill i and all buyers $\beta > \hat\beta$ strictly prefer skill i to skill i'.

Lemma 1. *(No Rationing) In any equilibrium, $B = S$.*

Proof. WLOG, suppose that $B \geq S$. Then, a buyer meets a seller with probability $\rho = S/B$, and a seller meets a buyer with probability 1. Therefore, the values satisfy:

$$\forall i : v_i = \rho \sum_{j \in J} y_j \left[m_{ij} \left(v_i + \frac{s_{ij}}{2} \right) + (1 - m_{ij}) v_i \right] + (1 - \rho) v_i - c \Rightarrow \sum_{j \in J} y_j m_{ij} s_{ij} = \frac{2c}{\rho}$$

$$\forall j : w_j = \sum_{i \in I} x_i \left[m_{ij} \left(w_j + \frac{s_{ij}}{2} \right) + (1 - m_{ij}) w_j \right] - c \Rightarrow \sum_{i \in I} x_i m_{ij} s_{ij} = 2c$$

Therefore, since $\sum_{i \in I} x_i = \sum_{j \in J} y_j = 1$:

$$\frac{2c}{\rho} = \sum_{i \in I} x_i \sum_{j \in J} y_j m_{ij} s_{ij} = \sum_{j \in J} y_j \left(\sum_{i \in I} x_i m_{ij} s_{ij} \right) = 2c \Rightarrow B = S$$

□

The next lemma states that, in equilibrium, the agents' values are increasing and the marginal values are bounded by the expected marginal productivity.

Lemma 2. *In any equilibrium,*

$$\frac{\sum_{j \in J} y_j m_{i'j} (g_{i'j} - g_{ij})}{\sum_{j \in J} y_j m_{i'j}} \geq v_{i'} - v_i \geq \frac{\sum_{j \in J} y_j m_{ij} (g_{i'j} - g_{ij})}{\sum_{j \in J} y_j m_{ij}} > 0, \ \forall i' > i$$

$$\frac{\sum_{i \in I} x_i m_{ij'} (g_{ij'} - g_{ij})}{\sum_{i \in I} x_i m_{ij'}} \geq w_{j'} - w_j \geq \frac{\sum_{i \in I} x_i m_{ij} (g_{ij'} - g_{ij})}{\sum_{i \in I} x_i m_{ij}} > 0, \ \forall j' > j$$

In particular, if $m_{ij} = 1, \forall i, j$, then the marginal value equals the expected marginal productivity: $v_{i'} - v_i = \sum_{j \in J} y_j (g_{i'j} - g_{ij})$ and $w_{j'} - w_j = \sum_{i \in I} x_i (g_{ij'} - g_{ij})$.

Proof. The Constant Surplus and Efficient Matching conditions imply that:

$$\sum_{j \in J} y_j m_{ij} s_{ij} = 2c = \sum_{j \in J} y_j m_{i'j} s_{i'j} \geq \sum_{j \in J} y_j m_{ij} s_{i'j}$$

Subtracting the RHS from the LHS, and normalizing:

$$v_{i'} - v_i \geq \frac{\sum_j y_j m_{ij} (g_{i'j} - g_{ij})}{\sum_j y_j m_{ij}} > 0$$

The upper bound is derived analogously by switching i and i'. □

Lemma 2 also implies that there is a uniform bound on marginal values: $\max_j g_{i'j} - g_{ij} \geq v_{i'} - v_i \geq \min_j g_{i'j} - g_{ij}$.

Remark 1. The Constant Surplus equations have two further implications: First, they determine the values for unchosen (measure 0) skills, and therefore we are not free to set those values arbitrarily (for instance, to minus infinity). Second, every agent has at least one partner with whom the surplus is positive. Furthermore, that partner is not of measure 0, which implies that there are no pathological equilibria where an agent searches forever.

Remark 2. If $\langle z, M, (v_i), (w_j)\rangle$ is an equilibrium, then so is $\langle z, M, (v_i+t), (w_j-t)\rangle$ for any transfer $t \in \mathbb{R}$. Therefore, there is at least one degree of freedom in the equilibrium values. We now show that there is in fact exactly one degree of freedom. This is because the marginal values, i.e. Δv_i, are uniquely pinned down by the investment decisions and a Constant Surplus equation imposes an additional condition on the value functions.

3 An Illustrative Example

The following example illustrates the interaction between matching and investment in the simplest possible setting. There are two skills, $I = J = \{0, 1\}$. Each agent can either invest and become skilled $i = j = 1$, or not invest and remain unskilled $i = j = 0$.

The cost of becoming skilled is the agent's type, where $\beta \sim U[a, d]$, $\sigma \sim U[a, d]$ and $a \geq 0$. The average cost (which is also the median) is $\mu = (a + d)/2 = 3/2$ and the length is $l = d - a$. The production function is: $G = \left[\begin{smallmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 3 \\ 3 & 4 \end{smallmatrix}\right]$. In words, the match output of two skilled agents is $g_{11} = 4$, the match output of skilled-unskilled pairs is $g_{10} = g_{01} = 3$, and the match output of two unskilled agents is $g_{00} = 1$. This production function is *submodular* because marginal productivity is greater when an agent is matched to an unskilled agent than when matched to a skilled agent, $g_{01} - g_{00} = 2 > 1 = g_{11} - g_{10}$.

First Best. The first-best allocation provides a useful benchmark. Consider a social planner who controls the agents' investment decisions and frictionlessly matches the agents in order to maximize per-period total welfare (total output minus total investment cost). The planner would have agents with below-average costs invest and become skilled, and match agents negatively assortative (skilled-unskilled pairs).[7] Total per-period welfare is:

$$W^{FB} = 3 - \int_a^\mu \beta f(\beta) d\beta - \int_a^\mu \sigma f(\sigma) d\sigma = 3 - (3a + d)/4$$

That is, there is a measure 1 of pairs, each pair produces $g_{10} = g_{01} = 3$ units, and only the agents with below-average costs invest.

Constrained Efficiency. With costly random search, the planner's problem is to choose the agents' acceptance and investment decisions in order to maximize total per-period welfare (output minus investment and search costs), while respecting the steady-state conditions. The constrained-efficient allocation is spanned by the following three simple allocations.

1) **Negative Assortative Matching (NAM):** As in the first-best allocation, agents with below-average costs invest, those with above-average costs do not, and agents match negatively assortative. In the stock population, half of the buyers and sellers are skilled, and so each agent exits with probability $1/2$. The Inflow=Outflow equation (4) is $1/2 = N/4$, and thus the measure of buyers in the stock population is $N = 2$. Total per-period welfare is:

$$\mathcal{W}^{NAM} = g_{10} - 2Nc - \int_a^\mu \beta f(\beta) d\beta - \int_a^\mu \sigma f(\sigma) d\sigma = 3 - 4c - (3a + d)/4$$

That is, in every period, a unit measure of pairs exit and produce $g_{10} = g_{01} = 3$ units, the stock population incurs the search cost c, and the new-born agents with below-average costs invest.

[7] To see why this allocation is efficient, notice that the first-best allocation cannot have both skilled-skilled and unskilled-unskilled matches since then agents could be profitably rematched (because $g_{10} + g_{01} = 6 > 5 = g_{11} + g_{00}$). Suppose there is a positive measure of unskilled-unskilled matches. Less than half of the agents are skilled, and hence there is an agent with below-average investment cost who is unskilled. If she is matched with another unskilled agent, then it is efficiency-improving for her to invest (productivity gain of 2 and investment cost of less than $\mu = 3/2$). If she is matched with a skilled agent, then she can be swapped with an agent in an unskilled-unskilled pair, and then invest. Thus, all matches are skilled-unskilled and investment costs are minimized when all below-average cost agents invest and all others do not.

2) **All Skills Match:** The investment decisions are the same as in the previous allocation, but now agents accept any partner. Since the market clears every period, the stock population is $N = 1$, and half of the buyers and half of the sellers are skilled. Total per-period welfare is:

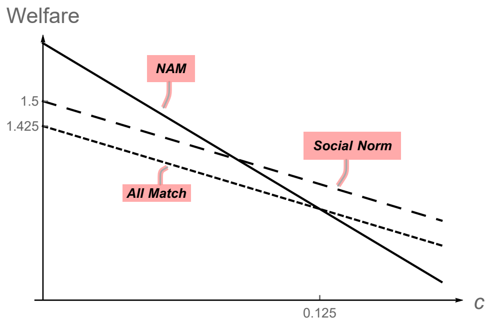
$$\begin{aligned}\mathcal{W}^{All} &= \frac{1}{4}(g_{11} + g_{10} + g_{01} + g_{00}) - 2Nc - \int_a^{\mu} \beta f(\beta) d\beta - \int_a^{\mu} \sigma f(\sigma) d\sigma \\ &= 2.75 - 2c - (3a + d)/4\end{aligned}$$

3) **Social Norm (one-sided investment):** Every buyer invests and becomes skilled and every seller remains unskilled. Agents accept any partner. Since the market clears in every period, the stock population is $N = 1$. The total per-period welfare is:

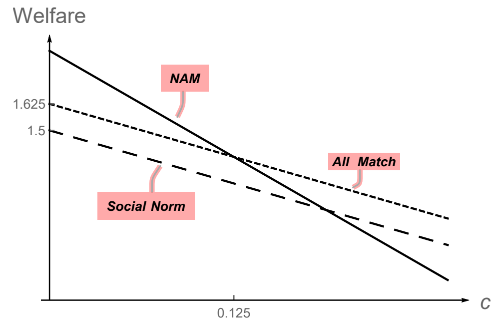
$$\mathcal{W}^{SN} = g_{10} - 2Nc - \int_a^d \beta f(\beta) d\beta = 3 - 2c - (a + d)/2$$

To sum up,

$$\begin{array}{rcccccc} \mathcal{W}^{NAM} & = & 3 & - & 4c & - & \left(\frac{3}{4}a + \frac{1}{4}d\right) \\ \mathcal{W}^{All} & = & 2.75 & - & 2c & - & \left(\frac{3}{4}a + \frac{1}{4}d\right) \\ \mathcal{W}^{SN} & = & 3 & - & 2c & - & \left(\frac{1}{2}a + \frac{1}{2}d\right) \end{array}$$



(a) Small Support, $l < 1$



(b) Large Support, $l > 1$

Figure 1: Welfare Comparison

Figure 1 depicts the welfare of each allocation as a function of the search cost c. The top and bottom panels illustrate the welfare ranking for cost distributions with small supports ($l < 1$) and large supports ($l > 1$), respectively.[8] These three

[8]The figure depicts the cases $l = 0.7$ and $l = 1.5$.

allocations demonstrate the trade-off between the three components of the welfare function: productivity, investment cost, and search cost. Each allocation optimizes two components at the expense of the third (see Table 1).

The NAM allocation maximizes productivity because the assignment is efficient and keeps the total investment cost low. The disadvantage is that every agent must search twice on average, in order to find the most productive partner.

The All Skills Match allocation reduces the search cost and keeps the total investment cost low. The disadvantage is mismatching, both unskilled-unskilled and skilled-skilled matches occur, which reduces total productivity. This allocation is better than the NAM allocation when the search cost is high, $c > 1/8$, and is worse when $c < 1/8$.

The Social Norm allocation maximizes productivity and minimizes the search cost, but the disadvantage is that buyers with above-average costs invest, while sellers with below-average costs do not, which increases the total investment cost. This talent misallocation problem is exacerbated as we stretch the support of the cost distribution. Therefore, the Social Norm allocation is better than the All Skills Match allocation when the support is small, $l < 1$, and is worse when $l > 1$.

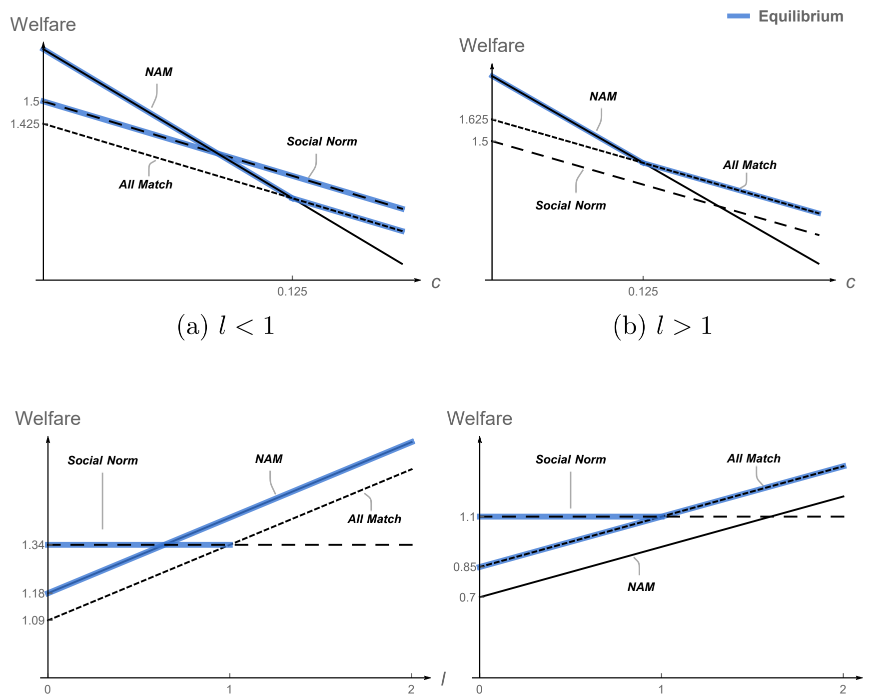
	Productivity	Search Cost	Investment Cost
NAM	✓	×	✓
All Skills Match	×	✓	✓
Social Norm	✓	✓	×

Table 1: Welfare Comparisons

The next claim establishes when each of these allocations is an equilibrium.

Claim 1. The NAM allocation is an equilibrium if and only if $c \leq 1/8$. The All Skills Match allocation is an equilibrium if and only if $c \geq 1/8$. The Social Norm allocation is an equilibrium if and only if $l \leq 1$.

The proof can be found in the Online Appendix. Figure 2 illustrates the equilibria and their welfare. Panels (a) and (b) depict welfare as a function of search cost c for the small and large support case (e.g. for $l = 0.7$ and $l = 1.5$). Panels (c) and (d) depict welfare as a function of the length l for low and high search costs,



(a) $l < 1$

(b) $l > 1$

(c) $c < 1/8$

(d) $c > 1/8$

Figure 2: Equilibrium Regions

respectively (e.g. for $c = 0.08$ and $c = 0.2$). The equilibrium regions are shaded blue. The key takeaways are:

1) The Second Welfare Theorem. The constrained efficient allocation, which is the upper envelope of the three lines, is an equilibrium allocation. In the next section, we establish a general second welfare theorem.

2) Assortative Matching. In all equilibria, there is negative assortative matching, which means that unskilled-skilled pairs always match. In Section 5.1, we establish assortative matching for any economy that has a super/submodular production function.

3) Discrimination. The efficient outcome can be discriminatory. Discrimination induces the two groups to invest differently and thereby minimizes search costs and enhances productivity, but at the expense of higher investment costs.

4) Multiplicity. Multiple equilibria may exist and the agents may fail to coordinate on the efficient one.

4 The Second Welfare Theorem

To simplify notation, we label the skills as $I = \{0, 1, \ldots, |I| - 1\}$ and $J = \{0, 1, \ldots, |J| - 1\}$. The constrained efficient allocation is the solution to the problem of a social planner who chooses the investment and acceptance strategies and sets the stock in the matching market, in order to maximize per-period total welfare, subject to the condition that the economy is in a steady state. Without loss of generality: i) the planner chooses a balanced state,[9] $B = S = N$; ii) the matching strategies are represented by a matching matrix; and iii) since the investment cost functions satisfy strictly increasing differences, the planners' optimal investment strategies can be defined by thresholds $\beta_0 \geq \beta_1 \geq \ldots \geq \beta_I$ and $\sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_J$, so that all buyers of type $\beta \in (\beta_{i+1}, \beta_i)$ choose skill i and all sellers of type $\sigma \in (\sigma_{j+1}, \sigma_j)$ choose skill j. Notice that the thresholds are descending because costs increase with type, so higher types choose lower skills. The planner chooses a tuple $\langle z, M, (\beta_i), (\sigma_j)\rangle$ of steady state, matching matrix, and investment thresholds in order to maximize:

$$\begin{aligned}\mathcal{W}\left(\langle z, M, (\beta_i), (\sigma_j)\rangle\right) = &\sum_{i\in I}\sum_{j\in J} N x_i y_j m_{ij} g_{ij} - 2Nc - \sum_{i\in I}\int_{\beta_{i+1}}^{\beta_i} C^b(i,\beta) f^b(\beta) d\beta \\ &- \sum_{j\in J}\int_{\sigma_{j+1}}^{\sigma_j} C^s(j,\sigma) f^s(\sigma) d\sigma\end{aligned} \tag{5}$$

subject to

$$\begin{aligned} flow_i^b &:= \left(F^b(\beta_i) - F^b(\beta_{i+1})\right) - N x_i \sum_{j\in J} y_j m_{ij} = 0, \forall i \\ flow_j^s &:= \left(F^s(\sigma_j) - F^s(\sigma_{j+1})\right) - N y_j \sum_{i\in I} x_i m_{ij} = 0, \forall j \\ x_i &\geq 0, \forall i \\ y_j &\geq 0, \forall j \\ X &:= 1 - \sum_{i\in I} x_i = 0 \\ Y &:= 1 - \sum_{j\in J} y_j = 0 \\ 1 &\geq m_{ij} \geq 0, \forall i, j \\ F^b(\beta_{|I|}) &= F^s(\sigma_{|J|}) = 0 \\ F^b(\beta_0) &= F^s(\sigma_0) = 1 \end{aligned}$$

[9] If $B > S$, then there exists another state with lower total search cost and identical output and investment cost.

The first term in the objective function is per-period total output (the measure of formed matches between buyer i and seller j is $Nx_iy_jm_{ij}$ and the match output is g_{ij}), the second term is the per-period total search cost, and the last two terms are the per-period total investment costs. The first constraint is that inflow equals outflow. The other conditions stipulate that x_i, y_j are proportions, m_{ij} are probabilities, and that the planner must assign a skill to every agent.

Remark 3. Notice that the maximization problem does not explicitly require that $\beta_0 \geq \beta_1 \geq \ldots \geq \beta_I$ and $\sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_J$, nor that $N > 0$, because these conditions are implied by the other constraints (see proof).

Theorem 1. *(Second Welfare Theorem) For every economy $\langle F^b, F^s, I, J, C^b, C^s, G, c\rangle$:*

i) There exists an optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$.

ii) Every optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$ can be decentralized. That is, there are values (v_i^*), (w_j^*), and a matching matrix M^* such that $\langle z, M^*, (v_i^*), (w_j^*)\rangle$ is an equilibrium, where $m_{ij}^* = m_{ij}$ for all i, j such that x_i, $y_j > 0$.

The theorem demonstrates that any optimum policy can be decentralized as an equilibrium. The proof shows that the equilibrium values that decentralize the optimal allocation are the shadow values of the flow constraints in the dual problem. We show that these values are internally self-consistent with the bargaining procedure, that is, they satisfy the Constant Surplus equations; and also motivate the agents to invest and match efficiently. For instance, if the planner wants buyer β and seller σ to choose skill i^* and j^*, then $i^* \in \arg\max_{i\in I} v_i - C^b(i, \beta)$ and $j^* \in \arg\max_{j\in J} w_j - C^s(j, \sigma)$; and if the planner wants them to accept (reject) each other, then $v_{i^*} + w_{j^*} \geq g_{i^*j^*}$ ($v_{i^*} + w_{j^*} \leq g_{i^*j^*}$).

Proof. First, we show that the constraints of the problem imply that $N > 0$, and $\beta_i \geq \beta_{i+1}$ for all i, and $\sigma_j \geq \sigma_{j+1}$ for all j. To see this, observe that $F^b(\beta_{|I|}) = 0$ and $F^b(\beta_0) = 1$, and so there exists a skill i such that $F(\beta_i) > F(\beta_{i+1})$. By constraint $flow_i^b$, it must be that $Nx_i \sum_{j\in J} y_jm_{ij} > 0$. Since x_i, y_j, m_{ij} are all non-negative, it follows that $N > 0$. Thus, the outflow of every skill is non-negative, and from the flow conditions, it must be that $\beta_i \geq \beta_{i+1}$ for all i, and likewise $\sigma_j \geq \sigma_{j+1}$ for all j. **(i)**

Existence: To demonstrate existence, since the objective is continuous, all we need to show is that the policy space is compact. First, there is a uniform upper bound $\overline{N}$ so that in any optimum, $N \leq \overline{N}$ (recall that $N \geq 0$). For the upper bound, notice that the Inflow=Outflow constraints imply $\sum_{i\in I}\sum_{j\in J} N x_i y_j m_{ij} = 1$, and therefore the first term in the welfare expression is a convex combination of g_{ij} and therefore is uniformly bounded by $\max g_{ij}$. Thus, $\lim_{N\to\infty} \mathcal{W} = -\infty$ and so the optimal policy cannot involve arbitrarily large N. The planner can choose quantiles $F(\beta_i)$ instead of thresholds β_i, and since the objective is also continuous in the quantiles and the quantile space is bounded, a maximum indeed exists.

(ii) Decentralizing optimal allocations: The dual problem is

$$\begin{aligned}
\mathcal{L}\left(\langle z, M, (\beta_i), (\sigma_j)\rangle\right) = &\sum_{i\in I}\sum_{j\in J} N x_i y_j m_{ij} g_{ij} - 2Nc \\
&- \sum_{i\in I}\int_{B_i} C^b(i,\beta) f^b(\beta) d\beta - \sum_{j\in J}\int_{S_j} C^s(j,\sigma) f^s(\sigma) d\sigma \\
&+ \sum_{i\in I} v_i \cdot flow_i^b + \sum_{j\in J} w_j \cdot flow_j^s + \sum_i \phi_i x_i + \sum_j \psi_j y_j + \gamma X + \lambda Y \\
&+ \sum_{i\in I}\sum_{j\in J} \left(\eta_{ij} m_{ij} + \hat{\eta}_{ij}(1 - m_{ij})\right)
\end{aligned}$$

We will first show that a constraint qualification holds and then construct an equilibrium using the shadow values from the KKT conditions.

1) The Constraint Qualifications: Since the problem is not convex, we use the constant rank regularity condition, which requires that for each subset of the gradients of the active inequality constraints and the equality constraints, the rank in the vicinity of the optimal point is constant (Janin [1984]). The formal proof is given in Lemma 4 in the Appendix.

2) Deriving values from the KKT conditions: Due to the constraint qualification above, the first order conditions (FOC) of the dual problem $\mathcal{L}$ are necessary at any optimum:

FOC(N):

$$\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} g_{ij} - 2c - \sum_{i\in I} v_i \left(x_i \sum_{j\in J} y_j m_{ij} \right) - \sum_{j\in J} w_j \left(y_j \sum_{i\in I} x_i m_{ij} \right) = 0$$

$$\iff \sum_i \sum_j x_i y_j m_{ij} (g_{ij} - v_i - w_j) = 2c$$

FOC(x_i) and FOC(y_j):

$$N\sum_j y_j m_{ij} g_{ij} - v_i N \sum_j y_j m_{ij} - N\sum_j w_j m_{ij} y_j - \gamma + \phi_i = 0$$
$$\iff N\sum_j y_j m_{ij}\left(g_{ij} - v_i - w_j\right) = \gamma - \phi_i$$

$$N\sum_i x_i m_{ij} g_{ij} - N\sum_i v_i x_i m_{ij} - w_j N\sum_i m_{ij} x_i - \lambda + \psi_j = 0$$
$$\iff N\sum_i x_i m_{ij}\left(g_{ij} - v_i - w_j\right) = \lambda - \psi_j$$

Complementary slackness: $\phi_i x_i = 0$ and $y_j \psi_j = 0$ and $\phi_i, \psi_j \geq 0$.

FOC(m_{ij}):

$$N x_i y_j g_{ij} - v_i N x_i y_j - w_j N x_i y_j + \eta_{ij} - \hat{\eta}_{ij} = 0$$
$$\iff N x_i y_j (g_{ij} - v_i - w_j) = -\eta_{ij} + \hat{\eta}_{ij}$$

Complementary slackness: $\eta_{ij} m_{ij} = 0$ and $\hat{\eta}_{ij}(1 - m_{ij}) = 0$ and $\eta_{ij}, \hat{\eta}_{ij} \geq 0$.

For $i \in \{1, \ldots, I-1\}$, FOC(β_i):

$$f^b(\beta_i)(v_i - v_{i-1}) = f^b(\beta_i)\left(C(i, \beta_i) - C(i-1, \beta_i)\right)$$

For $j \in \{1, \ldots, J-1\}$, FOC(σ_j):

$$f^s(\sigma_j)(w_j - w_{j-1}) = f^s(\sigma_j)\left(C(j, \sigma_j) - C(j-1, \sigma_j)\right)$$

We now show that the shadow values v_i, w_j, together with the matching matrix M and state z, constitute an equilibrium.

Decentralizating the constrained optimal allocation when z is interior (ii): To verify the Constant Surplus equations, notice that:

$$N \cdot 2c = N\sum_I \sum_J x_i y_j m_{ij}(g_{ij} - v_i - w_j) = \sum_I x_i N \sum_J y_j m_{ij}(g_{ij} - v_i - w_j)$$
$$= \sum_I x_i(\gamma + \phi_i) = \sum_I \gamma x_i + \phi_i x_i = \sum_I \gamma x_i = \gamma$$

The first line uses FOC(N), while the second line uses FOC(x_i), complementary slackness ($\phi_i x_i = 0$), and the condition $\sum_I x_i = 1$. Therefore $\gamma = 2cN$. Since z is interior, $\phi_i = 0$, and the FOC(x_i) is $\sum_J y_j m_{ij} (g_{ij} - v_i - w_j) = 2c$, which is the Constant Surplus equation for skill i. An analogous argument holds for the sellers' Constant Surplus equations.

To verify the Efficient Matching conditions, notice that if $g_{ij} - v_i - w_j > 0$, the FOC for m_{ij} requires that $\hat{\eta}_{ij} > 0$ and therefore $m_{ij} = 1$. Similarly, if $g_{ij} - v_i - w_j < 0$, the FOC for m_{ij} requires that $\eta_{ij} > 0$ and therefore $m_{ij} = 0$.

To verify that the investments are incentive compatible, we show that for any type $\beta \in [\beta_{i+1}, \beta_i]$, their most preferred skill is i. To see this, for any lower skill, $i' \le i$, the FOC for the threshold $\beta_{i'}$ is $f(\beta_{i'})(v_{i'} - v_{i'-1}) = f(\beta_{i'})(C(i', \beta_{i'}) - C(i'-1, \beta_{i'}))$ and recall that $\beta_{i'} \ge \beta$. Since $f > 0$ everywhere, this can be simplified to $v_{i'} - C(i', \beta_{i'}) = v_{i'-1} - C(i'-1, \beta_{i'})$. Since type $\beta_{i'}$ is indifferent between the skills i' and $i'-1$, by single-crossing, type β weakly prefers skill i' to skill $i'-1$. Thus, type β weakly prefers i to any lower skill i'. An analogous argument applies for higher skills.

The case of a non-interior z can be found in the Appendix (Section 8.1). □

It immediately follows from Theorem 1 that an equilibrium exists.

Corollary 1. *An equilibrium exists.*

The following proposition demonstrates some comparative statics for welfare.

Proposition 1. *The welfare function $\mathcal{W}$ is continuous, strictly decreasing, and convex in c. Moreover, the population size N is weakly decreasing in c.*

The proof is in the Appendix. It relies on the observation that $\partial\mathcal{W}/\partial c = -2N$, which follows immediately from the envelope theorem, implying that a shock to c has greater impact on welfare when c is small than when c is large.

Remark 4. (Matching and Values of Unrealized Skills) Theorem 1 proves that any optimum can be decentralized (modulo matching between unrealized skills). The planner can match unrealized types in any fashion because they have no impact on welfare, and thus the optimization problem places no restriction on their matching. However, the equilibrium conditions (the Constant Surplus equations and Efficient Matching conditions) apply for all skills, including unrealized ones. In the Appendix (Section 8.1), we construct the matching and values for these unrealized skills.

5 Equilibrium Structure and Applications

In the previous section, we demonstrated the second welfare theorem: the constrained efficient allocation can be decentralized as an equilibrium. However, the first welfare theorem need not hold since the economy can have multiple equilibria and the agents may fail to coordinate on the efficient one.[10]

In this section, we show that it is not the case that "anything goes" and the equilibria have a clear and simple structure: Section 5.1 shows that every equilibrium exhibits assortative matching if the production function is super/submodular. Section 5.2 considers an additively separable production function (product market) and shows that the equilibrium is unique. Finally, Section 5.3 characterizes the equilibria of a symmetric two-skill economy (marriage market). These sections demonstrate the applicability of our model and its implications. Furthermore, they show that for our second welfare theorem, the efficient allocation is not caught in a widely cast net. All results in this section are proven in the Appendix.

5.1 Assortative Matching

Denote the matching set of skill-i buyers by $M_i = \{j : m_{ij} > 0\} \subseteq J$, this is the set of seller skills with whom buyer i matches. Similarly, for sellers, $M_j = \{i : m_{ij} > 0\} \subseteq I$. The maxima and minima of these sets are denoted $\overline{m}_i = \max M_i$, $\underline{m}_i = \min M_i$, $\overline{m}_j = \max M_j$ and $\underline{m}_j = \min M_j$. We say that a seller's matching set M_j is convex if $\underline{m}_j < i < \overline{m}_j$ implies that $m_{ij} = 1$, and we define convexity similarly for a buyer. A matching matrix M exhibits *positive assortative matching* (PAM) if the matching sets are convex and the maxima/minima are weakly increasing. Likewise, M exhibits *negative assortative matching* (NAM) if the matching sets are convex and the maxima/minima are weakly decreasing. Finally, we say that *All Skills Match* if $m_{ij} = 1$ for all i, j.

The production function G is *supermodular (submodular)* if the marginal productivity of every skill i, $g_{(i+1)j} - g_{ij}$, is strictly increasing (decreasing) in j, and the marginal productivity of every skill j, $g_{i(j+1)} - g_{ij}$, is strictly increasing (decreasing) in i; G is *separable* if the marginal productivity of every skill i is constant j, and the marginal productivity of every skill j is constant in i.

[10] Similar coordination issues also create multiplicity in the frictionless market (e.g. Noldeke and Samuelson [2015]).

Previous work established sufficient conditions for positive/negative assortative matching in a search-matching equilibrium within a single population of agents (Shimer and Smith [2001], Atakan [2006]). The case of matching between two symmetric populations is similar to a single population model as every agent faces the same matching problem. However, in many two-sided matching setting, such as labor and product markets, the two sides are inherently different. An open question in the literature is whether assortative matching holds when the two populations are not identical. The next result answers a firm yes. To our knowledge, this is the first paper which establishes assortativity (convexity + monotonicity) beyond the symmetric framework.

Theorem 2. *In equilibrium, there is PAM whenever G is supermodular, NAM whenever G is submodular, and All Skills Match whenever G is separable.*

To outline the argument, we first show that the surplus function s_{ij} inherits super/submodularity from G. We use this observation and Lemma 2 to establish that the bounds of the matching sets are monotonic. We prove convexity from algebraic manipulations of the Constant Surplus equations. In contrast, existing proofs rely heavily on symmetry (Shimer and Smith 2001; Atakan 2006). In the discounting case, to show that the matching sets are convex, Shimer and Smith [2001] place further restriction on the production function which imply that the surplus function s_{ij} is convex. Our proof works for any super/submodular production function.[11]

m_{ij}	j_1	j_2	j_3	j_4	j_5
i_1					
i_2					
i_3					
i_4					
i_5					

Table 2: A PAM matrix: $m_{ij} = 1$ (blue), $0 < m_{ij} < 1$ (green), and $m_{ij} = 0$ (blank)

In Table 2, we depict a matching matrix that satisfies PAM. To maintain PAM, this matrix cannot be modified so that buyer 1 matches with seller 3 (pure or mixed) because that would violate the convexity condition for buyer 1. Likewise, it cannot be that buyer 2 matches with seller 5 because that would violate monotonicity.

[11]In fact, there are examples where G is supermodular and s_{ij} is not convex, and yet there is PAM.

Remark 5. The assortative matching result is useful for numerical analysis. For example, in the 5×5 case depicted above, there are $2^{25} \approx 33.6$ million pure matching matrices, but only $2,762$ of them satisfy PAM. In the 5×7 case, there are $2^{35} \approx 34$ trillion pure matching matrices, of which only $21,659$ satisfy PAM.[12]

Having analyzed the matching sets, we proceed to the investment decisions.

5.2 A Product Market (Separable Production)

We consider a product market in which each seller can produce one unit of a homogeneous good and each buyer desires a single unit. A buyer that invests in skill i receives the payoff α_i from consuming the good and a seller that invests in skill j can produce the good at a cost κ_j. The consumption value α_i is increasing in i and the cost κ_j is decreasing in j. When a buyer and seller meet, their output is $g_{ij} = \alpha_i - \kappa_j$. This production function is separable because the marginal productivity $g_{i'j} - g_{ij}$ is independent of j. In general, any separable production function can be written as $g_{ij} = g_i + g_j$, and so the following analysis applies for any separable production function.

As in Gale [1987], we allow for endogenous entry, that is, [13] every new-born agent either invests and enters the market or opts out and receives the outside payoff equal to u^b for buyers and u^s for sellers.

Proposition 2. *Any economy with a separable production function (with or without outside options) has a* ***unique*** *equilibrium and its allocation achieves the first best.*

Proof. The first-best allocation is unique and satisfies:

First-Best Matching: All pairs match. Since the marginal productivity of an agent is not affected by the skills of her partner, all pairs match to minimize the search cost.

First-Best Investment: Buyer β and seller σ acquire the skills: $i^*(\beta) = \arg\max_i \alpha_i - C^b(i, \beta)$ and $j^*(\sigma) = \arg\max_j -\kappa_j - C^s(j, \sigma)$. Denote by $C^{b*}(\beta) = C^b(i^*(\beta), \beta)$ the investment cost buyer β pays to acquire the efficient skill, and likewise $C^{s*}(\sigma) = C^s(j^*(\sigma), \sigma)$.

[12] At 1000 calculations per second, this is the difference between a program taking a millennium and 21 seconds.

[13] In Gale [1987], the output function is separable and the agents make entry decisions, but there are no investments. In Section 6, we allow for endogenous entry and show that our second welfare theorem is robust.

The social welfare of a match between buyer β and seller σ is $\omega(\beta,\sigma) = \alpha_{i^*(\beta)} - C^{b*}(\beta) - \kappa_{j^*(\sigma)} - C^{s*}(\sigma) - 2c$. To focus on the interesting case, we assume that there are types, $\beta', \sigma', \hat{\beta}, \hat{\sigma}$ such that $\omega(\beta',\sigma') > u^b + u^s > \omega(\hat{\beta},\hat{\sigma})$. This implies that in the first-best, some agents enter and others don't.[14]

First-Best Entry: Buyer β and seller σ enter iff $\beta \leq \beta_0$ and $\sigma \leq \sigma_0$. The entry thresholds are pinned down by[15] $F^b(\beta_0) = F^s(\sigma_0)$ and $\omega(\beta_0,\sigma_0) = u^b + u^s$.

Since g is separable, Lemma 2 implies that in equilibrium, the marginal value equal the marginal productivity: $\Delta v_i = \alpha_{i+1} - \alpha_i$, for every i, and $\Delta w_j = -(\kappa_{j+1} - \kappa_j)$, for every j. Therefore, the match surplus $s_{ij} = \alpha_i - \kappa_j - v_i - w_j$ is constant. As a result:

Equilibrium Matching: Theorem 2 demonstrates that in every equilibrium, all skills match.

Equilibrium Investment: The individually optimal investments satisfy

$$\arg\max_i \left\{v_i - C^b(i,\beta)\right\} = \arg\max_i \left\{\alpha_i - C^b(i,\beta)\right\}, \text{ for evey } \beta$$
$$\arg\max_j \left\{w_j - C^s(j,\sigma)\right\} = \arg\max_j \left\{-\kappa_j - C^s(j,\sigma)\right\}, \text{ for every } \sigma$$

The maximizers are equal because $\alpha_i - v_i$ and $-\kappa_j - w_j$ are constant

Equilibrium Entry: First, we show that there is entry. If not, then $v_{i^*(\beta)} - C^{b*}(\beta) \leq u^b$ and $w_{j^*(\sigma)} - C^{s*}(\sigma) \leq u^s$, for all β, σ, and so $v_{i^*(\beta)} - C^{b*}(\beta) + w_{j^*(\sigma)} - C^{s*}(\sigma) \leq u^b + u^s$. Substituting in the Constant Surplus equations, it follows that, $\alpha_{i^*(\beta)} - C^{b*}(\beta) - \kappa_{j^*(\sigma)} - C^{s*}(\sigma) - 2c \leq u^b + u^s$, which violates the assumption that there are types, β', σ' such that $\omega(\beta',\sigma') > u^b + u^s$. By a similar argument, it cannot be that all agents enter. Second, since some agents enter and others do not, denote by $\underline{\beta}, \underline{\sigma}$ the threshold types for whom the entry constraints hold with equality, notice that

$$\begin{aligned} u^b + u^s &= v_{i^*(\underline{\beta})} - C^{b*}\left(\underline{\beta}\right) + w_{j^*(\underline{\sigma})} - C^{s*}\left(\underline{\sigma}\right) \\ &= \alpha_{i^*(\underline{\beta})} - C^{b*}\left(\underline{\beta}\right) - \kappa_{j^*(\underline{\sigma})} - C^{s*}\left(\underline{\sigma}\right) - 2c = \omega(\underline{\beta},\underline{\sigma}) \end{aligned}$$

The second equality follows from the Constant Surplus equation, $v_i + w_j = \alpha_i - \kappa_j - 2c$.

[14] The case where everyone enters is trivial.

[15] Since buyers and sellers exit in equal numbers, in a steady state they must also enter in equal numbers.

In a steady state, the same measure of buyers and sellers enter, $F^b(\underline{\beta}) = F^s(\underline{\sigma})$. These two equations are the same as the equations that characterized the first-best entry decisions, and therefore it must be that $\underline{\beta} = \beta_0$ and $\underline{\sigma} = \sigma_0$. $\square$

Theorem 2 demonstrates that in any equilibrium, All Skills Match. The rest of the proof immediately follows from Lemma 2: when All Skills Match ($m_{ij} = 1$ for all i, j), in equilibrium, the marginal values equal marginal productivity, and hence the agents' private incentives to invest are aligned with the planner. Finally, **a law of one price** prevails (all trades occur at one price) and endogenous entry uniquely pins down the price that equates supply and demand.[16]

When investments by sellers affect the consumption value of buyers, the match output g_{ij} is no longer separable and market prices must also solve a non-trivial matching problem. The next subsection considers an economy with a nonseparable production function.

5.3 A Marriage Market

We now consider a two-skill economy in which the two sides are symmetric (a marriage market). That is, $I = J = \{0, 1\}$, $F^b = F^s = F$, and $g_{10} = g_{01}$. Each agent can invest and become *skilled* ($i = 1$) or remain *unskilled* ($i = 0$). It is without loss of generality that only skill 1 is costly and the cost is the agent's type,[17] that is, $C(\beta, i) = \beta i$ and $C(\sigma, j) = \sigma j$. We assume throughout that F has a large support: both $F(g_{10} - g_{00})$ and $F(g_{11} - g_{10})$ are strictly positive and less than 1. This implies that the equilibrium is interior (by Lemma 2).

We use the notation $x = x_1$ and $y = y_1$, and therefore $1 - x = x_0$ and $1 - y = y_0$. We denote the marginal values by $\Delta v = v_1 - v_0$ and $\Delta w = w_1 - w_0$. The difference in marginal productivities, $\Delta g = g_{11} - g_{10} - (g_{01} - g_{00})$, measures the modularity of production: G is supermodular if $\Delta g > 0$, submodular if $\Delta g < 0$, and separable if $\Delta g = 0$. We will denote the matching matrices $M = \left[\begin{smallmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{smallmatrix}\right]$ by $M^{PAM} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$; $M^{NAM} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$; and $M^{All} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$.

[16] The buyer's payoff from a match is $v_i + \frac{s_{ij}}{2} = v_i + c$ and his value of the good is α_i, so there is an implied price p_{ij} where $p_{ij} = \alpha_i - v_i - c$. The proof shows that $a_i - v_i$ is independent of i, and hence p_{ij} is independent of *both* i and j, and hence the law of one price prevails.

[17] Rescale each cost function to $\hat{C}(\beta, i) = C(\beta, i) - C(\beta, 0)$ and then relabel the cost types as $\hat{\beta} = \hat{C}(\beta, 1)$. Thus, skill 0 is free and the cost of skill 1 is each agent's type.

As demonstrated in Section 3, this economy can have symmetric and asymmetric equilibria. The following propositions identify conditions for the existence and efficiency of each type.

Proposition 3 (Symmetry)**.** *If G is supermodular, then every equilibrium is symmetric, that is, $x = y$ and $\Delta v = \Delta w$ and M is symmetric.*

For the next proposition, assume that G is submodular and F is centered: $g_{11} - g_{10} \leq med(F) \leq g_{10} - g_{00}$. The assumption implies that, for a majority of agents, it is not a strictly dominant strategy to invest or to not invest.

Proposition 4. *If G is submodular and F is centered, then for sufficiently small c, there is a unique equilibrium: it is symmetric and satisfies $x = y = 1/2$, $N = 2$, $\Delta v = \Delta w$, and $M = M^{NAM}$.*

The previous two propositions demonstrate that asymmetric equilibria can not occur when G is supermodular or when the search cost is small. The next Proposition demonstrates the converse: asymmetric equilibria exist, and are in fact efficient, when G is sufficiently submodular and the search cost is large. We first establish that for large c, all skills match.

Lemma 3. *For c sufficiently large, in every equilibrium $M = M^{All}$.*

Intuitively, if c is large enough, then every pair would rather match than continue searching. Recall that in any equilibrium where all skills match, Lemma 2 shows that the marginal value of each skill equals its marginal productivity:

$$\Delta v = y(g_{11} - g_{10}) + (1 - y)(g_{01} - g_{00})$$

$$\Delta w = x(g_{11} - g_{01}) + (1 - x)(g_{10} - g_{00})$$

Therefore, in a symmetric equilibrium, $F(\Delta v) = x = y = F(\Delta w)$, the two above equations reduce to $\Delta v = g_{10} - g_{00} + \Delta g F(\Delta v)$. Moreover, if $\Delta g < 0$ (submodularity), then there is a unique Δv which solves this equation and we denote the threshold type who invests by $\beta^{sym} = \Delta v$.[18] Consequently, there is a unique symmetric equilibrium which is defined by this threshold.

[18]Existence follows because $F(g_{10} - g_{00}) > 0$ and $F(g_{11} - g_{10}) < 1$. Uniqueness is straightforward since the LHS is increasing in Δv and the RHS is decreasing in Δv.

Proposition 5. *If G is submodular and $|\Delta g| f(\beta^{sym}) > 1$, then for sufficiently large c, the efficient equilibrium is asymmetric.*

The intuition underlying these propositions is that the agents' investments are affected by the market values and their cost. The symmetry in costs pushes towards symmetry in investments. For supermodular production functions, investments are strategic complements, i.e. the payoff of each buyer from investing increases with the proportion of skilled sellers, and similarly for each seller, which also pushes the investment levels towards each other. In contrast, when production is submodular, investments are strategic substitutes, which pushes investment levels away from each other, which can generate asymmetric equilibria.

The intuition for Proposition 5 is that when c is sufficiently large and G is submodular, there is a unique symmetric equilibrium. A planner can deviate from this symmetric allocation by increasing one side's investment and reducing the other's. The advantage of such an asymmetric investment policy is that it increases productivity (by reducing the mismatch between skills) while the disadvantage is that it increases total investment cost. The productivity improvement term is proportional to $|\Delta g|$ and total investment cost depends on the density f. When the former outweighs the latter, this deviation improves welfare and thus the efficient equilibrium is asymmetric.

We can summarize the main takeaways as follows:

Supermodularity $\vee$ Low Search Cost $\Rightarrow$ Every equilibrium is symmetric

Strong Submodularity $\wedge$ High Search Cost $\Rightarrow$ The efficient equilibrium is asymmetric

To illustrate this takeaway and the equilibrium structure, consider first a case where the production function is supermodular and the search cost is small. The following proposition derives the necessary and sufficient conditions for an equilibrium.

Proposition 6. *Let G be supermodular. For sufficiently small c, every equilibrium satisfies the following conditions: $M = M^{PAM}$, $x = y$, $N = \frac{1}{x^2+(1-x)^2}$, and*

$$\overbrace{F(\Delta v) = F\left(\frac{g_{11}}{2} - \frac{c}{x} - \left(\frac{g_{00}}{2} - \frac{c}{1-x}\right)\right)}^{inflow} = \overbrace{\frac{x^2}{x^2+(1-x)^2}}^{outflow} \tag{6}$$

Furthermore, these conditions are also sufficient: if z satisfies them, then there are values (v_i), (w_j) such that $\langle z, M^{PAM}, (v_i), (w_j)\rangle$ is an equilibrium.

In words, there is strict positive assortative matching and the fundamental equation (6) pins down the state variables x, which in turn determines $y, N, \Delta v, \Delta w$. For the Efficient Matching condition to hold, the search costs must be low enough so that skilled and unskilled agents do not want to match with each other, which is equivalent to the following (see proof):

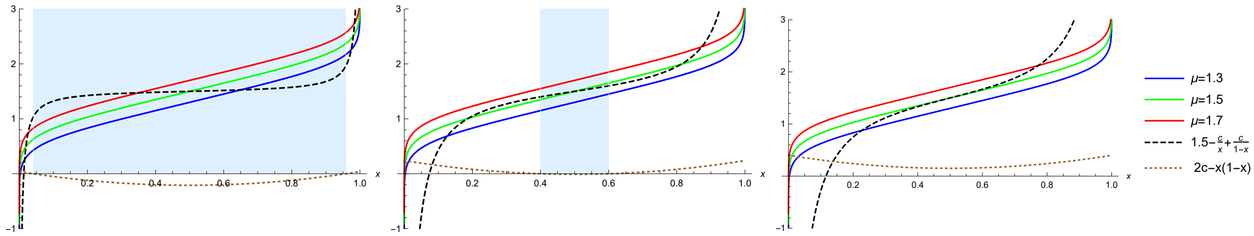
$$\Delta g x\,(1-x) \geq 2c \tag{7}$$

Equation (6) pins down precisely how the equilibria responds to an exogenous change in F or G. For example, consider $G^{sup} = \left[\begin{smallmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 4 \end{smallmatrix}\right]$, and $F = N(\mu, \rho^2)$ with the means $\mu = 1.3, 1.5, 1.7$ and standard deviation[19] $\rho = 0.3$. We rewrite equation (6) as:

$$1.5 - \frac{c}{x} + \frac{c}{1-x} = F^{-1}\left(\frac{x^2}{x^2+(1-x)^2}\right)$$

In Figure 3, we graph this equation: the dashed lines illustrate the LHS for different search costs, the solid lines illustrate the RHS for different distributions, and the intersections are the equilibrium candidates. Each candidate x is an equilibrium if and only if the Efficient Matching condition (7) holds, as depicted by the lightly shaded area. Notice that when $c = 0.02$, there is a unique PAM equilibrium for every μ (panel a). On the other hand, when search costs are $c = 0.2$, no equilibrium candidate satisfies the Efficient Matching condition (7) and a PAM equilibrium therefore does not exist (panel c). For $c = 0.12$, a PAM equilibrium exists when $\mu = 1.5$, but not when $\mu = 1.3$ or 1.7 (panel b).

[19] Notice that F has a large support, and therefore every equilibrium is interior.



(a) $c = 0.02$ (b) $c = 0.12$ (c) $c = 0.2$

Figure 3: PAM Equilibrium

Table 3 summarizes the equilibrium variables in the case of $c = 0.02$.

	$x = y$	N	$\Delta v = \Delta w$	$F(\Delta v)$	$Prod - Search - Inv = W$
$\mu = 1.7$	0.35	1.83	1.47	0.22	$1.66 - 0.07 - 0.58 = 1.02$
$\mu = 1.5$	0.5	2	1.5	0.5	$2.5 - 0.08 - 1.26 = 1.16$
$\mu = 1.3$	0.65	1.83	1.53	0.78	$3.34 - 0.07 - 1.83 = 1.44$

Table 3: Equilibrium Values, $c = 0.02$

In this example, when the search cost is high, a PAM equilibrium does not exist, but an All Skills Match equilibrium does. The following proposition provides the necessary and sufficient equilibria conditions for an All Skills Match equilibrium.

Proposition 7. *For sufficiently large c, every equilibrium satisfies the following conditions: $M = M^{All}$, $N = 1$, and*

$$x = F(\Delta v) = F\left(y(g_{11} - g_{10}) + (1 - y)(g_{01} - g_{00})\right) \tag{8}$$

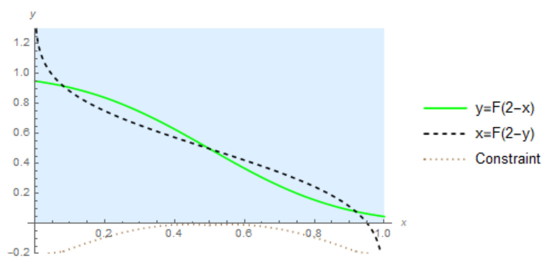
$$y = F(\Delta w) = F\left(x(g_{11} - g_{01}) + (1 - x)(g_{10} - g_{00})\right) \tag{9}$$

Furthermore, these conditions are also sufficient: if x, y satisfies (8) and (9), then there are values such that $\langle z, M^{All}, v, w\rangle$ is an equilibrium.

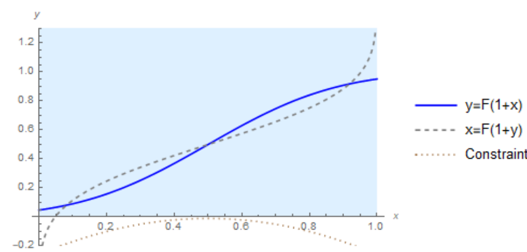
The fundamental equations (8) and (9) pin down the candidate state variables x, y that are consistent with the fact that the marginal value must equal the marginal cost (Lemma 2) and Inflow=Outflow. Each candidate (x, y) is an equilibrium if and only if the Efficient Matching conditions holds, so that every pair of agents would rather match than continue searching (see proof):

$$Sub: |\Delta g| \max\{(1-x)(1-y), xy\} \leq 2c \tag{10}$$
$$Sup: \Delta g \max\{(1-x)y, x(1-y)\} \leq 2c$$

For example, consider $c = 0.13$, $F = N(\mu, \rho^2)$ with $\mu = 1.5$ and $\rho = 0.3$, and two production matrices $G^{sup} = \left[\begin{smallmatrix}1 & 2\\ 2 & 4\end{smallmatrix}\right]$ and $G^{sub} = \left[\begin{smallmatrix}1 & 3\\ 3 & 4\end{smallmatrix}\right]$. In Figure 4, the dashed lines represent Equation (8), the solid lines denote Equation (9), and the intersection points are the equilibrium candidates. In panel (a), $G = G^{sub}$ and these equations reduce to $y = F(2-x)$ and $x = F(2-y)$; in panel (b), $G = G^{sup}$, and these equations reduce to $y = F(1+x)$ and $x = F(1+y)$. Every candidate (x, y) is an equilibrium if and only if it satisfies the Efficient Matching condition (10) as depicted by the shaded area.



(a) G^{sub}



(b) G^{sup}

Figure 4: All Match

Table 4 summarizes the equilibrium values for high c.

	Δv	Δw	$x = F(\Delta v)$	$y = F(\Delta w)$	$Prod - Search - Inv = W$
Symmetric	1.5	1.5	0.5	0.5	$2.75 - 2c - 1.26 = 1.49 - 2c$
Asymmetric 1	1.92	1.08	0.92	0.08	$2.93 - 2c - 1.41 = 1.52 - 2c$
Asymmetric 2	1.08	1.92	0.08	0.92	$2.93 - 2c - 1.41 = 1.52 - 2c$

Table 4: All Skills Match (G Submodular)

Notice that in panel (b), G is supermodular and every equilibrium is symmetric (i.e. the intersection points lie on the 45° line). However, in panel (a), G is submodular and there is one symmetric equilibrium, where $x = y = 1/2$, and two asymmetric ones, $x = 1 - y \approx 0.9$ and $x = 1 - y \approx 0.1$ (which are mirror images of each other). In Table 4, notice the tradeoff between investment cost and productivity: the asymmetric equilibrium achieves higher productivity than the symmetric one (2.93 vs 2.75) but total investment cost is also higher (1.41 vs 1.26). In this case, the former outweighs the latter, and thus the asymmetric equilibrium achieves higher welfare.[20]

Remark 6. Table 3 also illustrates the equilibrium effects of a uniform investment tax or subsidy. For example, imagine a government policy that pays a uniform subsidy of 0.2 to any agent who invests and thus shifts the cost distribution leftward from the red line ($\mu = 1.7$) to the green line ($\mu = 1.5$). As a result, the proportion of agents who invest more than doubles, from 22% to 50%, and the proportion of skilled agents in the stock population increases from 0.35 to 0.5. The subsidy increases inequality since skilled agents are made better off by the subsidy and it is now easier for them to find a partner (their expected search duration decreases from 2.86 to 2 periods), whereas unskilled agents are worse off because their search duration increases (from 1.54 to 2 periods) and they don't receive the subsidy. Total welfare increases by 0.14 (from 1.02 to 1.16), although importantly, the welfare gain does not offset the cost of the subsidy, which is 0.2 (the total inflow of buyers and sellers is 2 and half of them receive the subsidy).

6 Extensions

In this section, we discuss several extensions of the second welfare theorem.

6.1 Outside Options and Endogenous Entry

Suppose that every new-born agent can either invest and enter the market or opt out and receive the outside payoff equal to u^b for buyers and u^s for sellers. In equilibrium, buyer β enters the market if and only if $\max_i v_i - C(i, \beta) \geq u^b$, and seller σ enters if and only if $\max w_j - C(j, \sigma) \geq u^s$. We focus on the interesting case where there are gains to trade, and so for at least two types, β and σ, $\max_{i \in I, j \in J} g_{ij} - 2c - C(i, \beta) - C(j, \sigma) > u^b + u^s$.

The only difference from the baseline model is that the planner now also chooses the entry thresholds β_0 and σ_0 in order to maximize:

[20] Here, $|\Delta g| = 1$, and $f(\beta^{SYM}) = 1.32981$, so $|\Delta g| f(\beta^{SYM}) > 1$ and therefore the symmetric equilibrium is inefficient.

$$\mathcal{W} = N\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} g_{ij} - 2Nc - \sum_{i\in I}\int_{\beta_{i+1}}^{\beta_i} C(i,\beta) f^b(\beta)d\beta - \sum_{j\in J}\int_{\sigma_{j+1}}^{\sigma_j} C(j,\sigma) f^s(\sigma)d\sigma$$
$$+ \int_{\beta_0}^{\infty} u^b f^b(\beta)d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma)d\sigma$$

and the boundary conditions $F^b(\beta_0) = 1$ and $F^s(\sigma_0) = 1$ are removed.

Corollary 2. *In a model with outside options, the constrained efficient outcome is an equilibrium.*

The proof shows that the shadow values still constitute an equilibrium (see Appendix). As before, v_0 is the shadow value of the skill 0 flow constraint. However, there is an additional first-order condition since β_0 is now endogenous: $v_0 - C(0, \beta_0) = u^b$ which is precisely the equilibrium entry condition for buyers. An analogous argument holds for sellers.

Remark 7. In the baseline model, there is exactly one degree of freedom in the equilibrium values (see Remark 2). In the model with outside options, there is an additional entry condition and thus the values are unique.

6.2 Asymmetric Bargaining Weight and Search Costs

We now allow buyers and sellers to differ in their search costs, c^b and c^s, and their bargaining weights, α and $1 - \alpha$. When a buyer with skill i and a seller with skill j accept each other, the buyer receives $v_i + \alpha s_{ij}$ and the seller receives $w_j + (1 - \alpha)s_{ij}$. In the baseline model, Lemma 1 establishes that the equilibrium state is balanced, $B = S$, and the proof turned on the assumptions that $c^b = c^s = c$ and $\alpha = 1/2$. If either assumption does not hold, then the equilibrium state can be unbalanced, for instance, if $B > S$, then every buyer meets a seller with probability S/B and every seller always meets a buyer.

Corollary 3. *The constrained efficient allocation is an equilibrium if and only if* $\alpha = \frac{c^b}{c^s + c^b}$.

Proof. In the constrained efficient allocation, the state must always be balanced. Otherwise, WLOG, if $B > S$, then the planner can increase welfare by setting $B = S$ without affecting productivity or the investment cost.

Define $\mu = min(B, S)$. In equilibrium, the values satisfy:

$$v_i = (\mu/B)\left(\sum_{j\in J} y_j\left[m_{ij}\left(v_i + \alpha s_{ij}\right) + (1-m_{ij})v_i\right]\right) + (1-\mu/B)\, v_i - c^b, \forall i$$

$$w_j = (\mu/S)\left(\sum_{i\in I} x_i\left[m_{ij}\left(w_j + (1-\alpha)s_{ij}\right) + (1-m_{ij})w_j\right]\right) + (1-\mu/S)w_j - c^s, \forall j$$

Rewriting, we obtain the modified Constant Surplus equations:

$$\begin{aligned}\sum_{j\in J} y_j m_{ij} s_{ij} &= \frac{c^b}{\alpha\,(\mu/B)}, \forall i \\ \sum_{i\in I} x_i m_{ij} s_{ij} &= \frac{c^s}{(1-\alpha)\,(\mu/S)}, \forall j\end{aligned} \tag{11}$$

$$\Rightarrow \frac{c^b}{\alpha\,(\mu/B)} = \sum_{i\in I} x_i \sum_{j\in J} y_j m_{ij} s_{ij} = \sum_{j\in J} y_j \sum_{i\in I} x_i m_{ij} s_{ij} = \frac{c^s}{(1-\alpha)\,(\mu/S)}$$

$$\Rightarrow \frac{B}{S} = \frac{\alpha}{1-\alpha}\cdot\frac{c^s}{c^b} \tag{12}$$

Therefore, in equilibrium, the state is balanced if and only if $\alpha = \frac{c^b}{c^s+c^b}$. Since the constrained efficient allocation has to be balanced, this condition is necessary for efficiency. For sufficiency, suppose that $\alpha = \frac{c^b}{c^s+c^b}$. With this α, the Constant Surplus equations (11) are the same as in a model where buyers and sellers have the same search cost $c' = (c^s + c^b)/2$, and same bargaining weight $\alpha' = 1/2$. Moreover, the two models also identical constrained efficient allocations and equilibria. Therefore, applying our second welfare theorem to that model yields equilibrium values (v_i) and (w_j) that support the constrained efficient outcome as an equilibrium of this model. □

Remark 8. By Equation 12, if the bargaining weight $\alpha \neq \alpha^* \equiv \frac{c^b}{c^s+c^b}$, then any equilibrium is unbalanced and therefore inefficient. In particular, $B > S$ whenever $\alpha > \alpha^*$, and $B < S$ whenever $\alpha < \alpha^*$. Intuitively, the second welfare theorem holds when buyers' bargaining power equals their proportion of search costs.

Corollary 4. *For any bargaining weight* α*,* and search costs c^b and c^s, *an equilibrium exists. Moreover, in every equilibrium, there is PAM (NAM) whenever* G *is supermodular (submodular).*

6.3 Alternative Meeting Functions and the Hosios Condition

We now consider a general meeting function where $\mu(B, S)$ is the total number of meetings in a period. In every period, each agent can meet at most one other agent, and so $\mu(B, S) \leq \min\{B, S\}$. Meetings are still random and the probability that a buyer meets a seller is $\mu(B, S)/B$, while the probability that a seller meets a buyer is $\mu(B, S)/S$. As is standard, we take μ to be homogeneous of degree 1 and differentiable.

Corollary 5. *The constrained efficient allocation is an equilibrium if and only if*

$$\alpha = \frac{B^* c^b}{B^* c^b + S^* c^s} = \frac{\partial \mu(B^*, S^*)/\partial B}{\mu(B^*, S^*)/B^*}$$

where B^*, S^* *are the constrained efficient stock.*

In words, the constrained efficient allocation can be decentralized as an equilibrium if and only if the bargaining weight of each side equals their share of the overall search costs, which also equals the elasticity of the meeting function at the optimum (the Hosios condition). The proof closely follows that of the welfare theorem (see Appendix). Hosios (1990) shows that when agents are *homogeneous* the search externalities that they impose on each other are perfectly offset under the "right" sharing rule. In contrast, in our model, agents are heterogeneous and they make ex-ante investments. Remarkably, the same sharing rule still works.

6.4 Time Discounting

In our model, agents incur a fixed additive search cost in every period and do not discount time. In a model with time discounting, the constrained efficient allocation is not an equilibrium outcome (generically). The key issue is that when two agents reject each other, each one incurs an implicit search cost because her payoff is delayed. These implicit search costs depend on an agent's value, and therefore acquiring a higher skill entails acquiring a higher implicit search cost, which perturbs the incentives to invest.

This link between agents' values and search costs can also distort the matches that form in equilibrium. In contrast, a fixed additive search cost severs this link.

To illustrate, consider the case of two skills $I = J = \{0, 1\}$; separable production, $g_{ij} = g_i + g_j$; equal bargaining weights, $\alpha = 1/2$; and a discount rate of $\delta \in [0, 1]$. For any equilibrium of the time discounting model, the equilibrium values must satisfy the standard conditions:

Value Equations	Efficient Matching
$v_i = \sum_{j \in J} y_j m_{ij} \frac{1}{2}(g_{ij} - \delta w_j + \delta v_i)$ $w_j = \sum_{i \in I} x_i m_{ij} \frac{1}{2}(g_{ij} - \delta v_i + \delta w_j)$	$m_{ij} = \begin{cases} 1 & \text{when } \delta(v_i + w_j) < g_{ij} \\ 0 & \text{when } \delta(v_i + w_j) > g_{ij} \end{cases}$

Since production is separable, the efficient matching rule is All Skills Match, for any discount factor. Plugging $m_{ij} = 1, \forall i, j$ into the above equations and differencing, we get: $\Delta w = \Delta v = \frac{g_1 - g_0}{2 - \delta}$.

Case 1. Inefficient investment. If All Skills Match, then the efficient investment rule is that all buyers and sellers with costs below $g_1 - g_0$ will invest. However, the above equations show that only buyers and sellers with costs below $\frac{g_1 - g_0}{2 - \delta}$ invest, and therefore, there is underinvestment in equilibrium. Intuitively, this is because when an agent invests, they increase their value and also their implicit search cost, which dampens the investment incentive.

$\Delta v = \Delta g - \Delta v(1 - \delta)$, $\Delta v(2 - \delta) = \Delta g$, $\Delta v = \frac{\Delta g}{2 - \delta}$

This can be seen by rewriting the above value equations in a similar style to our Constant Surplus Equations where the implicit search costs are $(1 - \delta)v_i$ for buyers and $(1 - \delta)w_j$ for sellers.

$$2(1 - \delta)v_i = \sum_{j \in J} y_j m_{ij}(g_{ij} - \delta w_j - \delta v_i)$$

$$2(1 - \delta)w_j = \sum_{i \in I} x_i m_{ij}(g_{ij} - \delta v_i - \delta w_j)$$

Case 2. Inefficient matching. We now show that All Skills Match need not be supported as an equilibrium. Since buyers and sellers have the same marginal values, the equilibrium is symmetric, $x = y = F(\Delta v)$, and WLOG $v_0 = w_0$, $v_1 = w_1$.[21]

[21] Notice that given any symmetric All Skills match equilibrium with values $(\hat{v}_0, \hat{v}_1, \hat{w}_0, \hat{w}_1)$, there is an equivalent equilibrium where $v_0 = w_0 = (\hat{v}_0 + \hat{w}_0)/2$ and $v_1 = w_1 = (\hat{v}_1 + \hat{w}_1)/2$.

Therefore, the unskilled buyer's value is $v_0 = \frac{1}{2}y(g_{10} - \delta \Delta v) + \frac{1}{2}(1-y)g_{00}$. Substituting in from the above equations, we obtain that $v_0 - \delta g_0 > 0$ if and only if

$$\frac{g_1}{g_0} > \frac{1}{y}\frac{2-\delta}{\delta} + 1$$

That is, low-skill agents will not match with each other whenever g_1/g_0 is too large. Intuitively, this is because unskilled agents have low search costs and therefore "hunt" for skilled agents from whom they can extract more surplus (because the skilled agents have higher implicit search costs). For example, if $y \geq 0.9$, $\delta \geq 0.9$, and $g_0 = 1$, then unskilled agents won't match if $g_1 > 2.36$.

Remark 9. Under random search and bargaining with time discounting, it is often found that the constrained efficient allocation is approximated by an equilibrium as the discount factor goes to 1 (Gale, 1987). However, when the discount factor is uniformly bounded away from 1, the equilibrium is typically not constrained efficient (Acemoglu and Shimer 1999a, Shimer and Smith 2001, Mortensen and Wright 2002).

7 Discussion

We developed a model to study the structure and efficiency of equilibria in two-sided matching markets with heterogeneous agents. The key features of the market are that the agents invest in skills before entering and the trading process is decentralized: agents must engage in costly search and bargain pairwise over the joint output.

There are three main points. First, the constrained efficient allocation is an equilibrium. That is, there are values that satisfy the equilibrium conditions and also perfectly align the individual incentives to invest and to accept/reject matches with the planner's. The welfare theorem also guarantees the existence of equilibrium. Second, the equilibria have a clear structure. Strikingly, we establish a general assortative matching result for two-sided markets with super/submodular production functions. Moreover, if the production function is additively separable, then there exists a unique equilibrium. For nonseparable production functions, we characterized the equilibria of two-skill symmetric economies and demonstrated some comparative statics. The equilibrium set is tractable and small, dispelling any concerns that the welfare theorem operates by catching the efficient outcome in a "widely cast net". Third, for this symmetric economy, we showed that if there is strong submodularity and a substantial search cost, then the efficient equilibrium is asymmetric.

The welfare and sorting results are robust in the sense that they continue to hold in economies with outside options and for meeting functions that satisfy constant returns to scale (see Section 6). However, the welfare theorem relies on the assumptions that buyers and sellers incur the same additive search costs and have the same bargaining weights. More generally, if the two sides have different search costs, then the constrained efficient allocation is an equilibrium if and only if the bargaining weight $\alpha = \frac{c^b}{c^b+c^s}$ (see Corollary 3). This condition is equivalent to the famous Hosios condition (Hosios, 1990). Otherwise, the stock variable is inefficient in every equilibrium. For example, if $\alpha > \frac{c^b}{c^b+c^s}$, then buyers are in an advantageous position and become the long side of the market, $B > S$. Every unbalanced stock variable is inefficient, and there may be additional distortions of investment and matching.

However, when agents discount time, each agent's search cost is implicitly caused by delaying payoffs and it is therefore proportional to their continuation value in the market. This link between search costs and the value of each skill in the market has consequences for efficiency and sorting in discounting models. First, when an agent acquires a higher skill, she also acquires a higher search cost which reduces her incentive to invest (see Subsection 6.4). Second, it may be impossible for agents to match efficiently in any equilibrium. We provided an example where unskilled agents do not match with each other, but rather hunt skilled agents from whom they can extract surplus (skilled agents are vulnerable because of their higher implicit search costs, see Subsection 6.4). Third, sorting results require strong assumptions on the production function (see Shimer and Smith, 2000). We show that by severing this link, the fixed additive search model delivers powerful results: the Second Welfare Theorem and the sorting result are very general, existence is guarantee, and the equilibria have a clear and intuitive structure.

Our results have several implications:

Labor Market - Our results address a central question in the literature on sorting in the labor market: When do high-skill workers match with high-tech (or capital intensive) firms? The previous sorting results of Shimer and Smith [2000] and Atakan [2006] do not apply to the labor market because the agents on opposite sides do not face the same type distribution. Our results provide a theoretical foundation for assortative matching (both monotonic and convex matching sets) for two-sided markets.

There is an extensive literature studying the mismatch between labor skills and production technologies. However, the skill-technology mismatch also affects (and is affected by) investment in human and physical capital. For instance, a lower search cost generally leads to finer sorting, which affects the marginal productivity of some skills and thereby the incentive to invest. Alternatively, a change in the investment costs changes the composition of skills in the market, which may further impact search and matching. Our model provides a general framework to study investment and matching together, rather than in isolation, including how they are affected by agents' search cost, investment cost, and production function.

Product and Marriage Markets - The joint production function in a product market is often assumed to be separable. Therefore the equilibrium is unique, and by our Welfare Theorem, it must be efficient. On the other hand, in a marriage market, the joint household production function typically has complementarities between skills, which can generate multiple equilibria. It is not surprising that a symmetric economy has symmetric equilibria, but we show that there can also be asymmetric equilibrium, which can even be efficient. The asymmetric equilibrium is discriminatory in the sense that the return on investment depends on gender, which generates a gap in skill acquisition. This gender gap can persist even when it is inefficient (see Section 3) and in some cases can be corrected by a policy intervention such as an investment subsidy or tax.

Taxes and Subsidies - It immediately follows from our second welfare theorem that at the efficient equilibrium, any policy intervention causes a harmful distortion (see Remark 6 in Subsection 5.3). However, there is still room for policy interventions at inefficient equilibria. For example, an investment subsidy in the marriage market can boost welfare by eliminating inefficient discriminatory equilibria without affecting the efficient one. To illustrate, consider the example in Section 3 depicted in Figure 2: in the regions where the Social Norm equilibrium is inefficient (panels a and c), a finely tuned investment tax or subsidy can steer the economy away from the Social Norm equilibrium towards the efficient NAM equilibrium.[22] In the regions where the NAM or All Skills Match equilibria are inefficient (panels a,c,d), a policy that subsidizes investment only for one side of the market can uniquely implement the efficient Social Norms equilibrium.

[22]If the government pays a subsidy t to every agent who invests, and $a-1 < t < d-1$, then the subsidy eliminates the SN equilibria without impacting the NAM equilibrium (which is the only remaining one).

Applications and Simulations: In the two-skill economy, a full characterization of equilibria and comparative statics can be done analytically. However, this analytical approach is more difficult in larger economies. The welfare and sorting results are useful technical tools for applying and simulating the model. In particular, the planner's problem is more amenable to numerical simulations than the equilibrium conditions, as there are less conditions and values need not be derived. For an n-skill economy, the endogenous variables $(v_i), (w_j), (x_i), (y_j), (\beta_i), (\sigma_j)$ are of order n, but the matching matrix $[m_{ij}]$ is of order n^2. The assortative matching result facilitates simulations by reducing the number of matching variables (from n^2 to $2n$), which brings the whole problem from $O(n^2)$ to $O(n)$. It remains to be seen whether the model can be calibrated to derive useful empirical predictions, but the theoretical results found here are promising.

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8 Online Appendix

8.1 Omitted Proofs for Section 4

Theorem 1. *(Second Welfare Theorem) For every economy $\langle F^b, F^s, I, J, C^b, C^s, G, c\rangle$:*

i) There exists an optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$.

ii) Every optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$ can be decentralized. That is, there are values (v_i^*), (w_j^*), and a matching matrix M^* such that $\langle z, M^*, (v_i^*), (w_j^*)\rangle$ is an equilibrium, where $m_{ij}^* = m_{ij}$ for all i, j such that x_i, $y_j > 0$.

We now prove the non-interior case; and the constraint qualification.

Proof. z **is non-interior:**

Given any optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$, the FOCs imply that there are shadow values (v_i), (w_j) such that (see proof of Theorem 1 in text):

$$\sum_j y_j m_{ij} \left(g_{ij} - v_i - w_j\right) \geq 2c \text{ with equality when } x_i > 0$$

$$\sum_i x_i m_{ij} \left(g_{ij} - v_i - w_j\right) \geq 2c \text{ with equality when } y_j > 0$$

$$N x_i y_j (g_{ij} - v_i - w_j) = -\eta_{ij} + \hat{\eta}_{ij}$$

where $\eta_{ij} m_{ij} = 0$ and $\hat{\eta}_{ij}(1 - m_{ij}) = 0$ and $\eta_{ij}, \hat{\eta}_{ij} \geq 0$.

The above equations demonstrate the Constant Surplus equations for all i where $x_i > 0$. But, the Constant Surplus equation may not hold for skills i where $x_i = 0$. Therefore, for any skill i where $x_i = 0$, we define v_i^* to be the unique value which solves $\sum_j y_j \max\left\{g_{ij} - v_i^* - w_j, 0\right\} = 2c$. For any skill i where $x_i > 0$, we define $v_i^* = v_i$. Likewise, for sellers j where $y_j = 0$, define w_j^* to be the unique value which solves $\sum_j y_j \max\left(g_{ij} - v_i - w_j^*, 0\right) = 2c$ $y_j > 0$. For sellers j where $y_j > 0$, define $w_j^* = w_j$. Define a matching matrix by $m_{ij}^* = \mathbf{1}_{g_{ij} - v_i^* - w_j^* > 0}$ whenever $x_i = 0$ or $y_j = 0$ and setting $m_{ij}^* = m_{ij}$ otherwise.

It now remains to be seen that $\langle z, M^*, (v_i^*), (w_j^*)\rangle$ satisfies the equilibrium constraints.

The Constant Surplus Equations hold: For any skill i where $x_i > 0$, from the above, we have that $\sum_j y_j m_{ij}^* \left(g_{ij} - v_i^* - w_j^*\right) = \sum_j y_j m_{ij} \left(g_{ij} - v_i - w_j\right) = 2c$ because $v_i^* = v_i$ and whenever $y_j > 0$, then $m_{ij} = m_{ij}^*$ and $w_j = w_j^*$. For any skill i

where $x_i = 0$,

$$\sum_j y_j m_{ij}^* \left(g_{ij} - v_i^* - w_j^*\right) = \sum_j y_j \max\left(g_{ij} - v_i^* - w_j^*, 0\right)$$
$$= \sum_j y_j \max\left(g_{ij} - v_i^* - w_j, 0\right) = 2c$$

because $w_j^* = w_j$ whenever $y_j > 0$. The same argument demonstrates the Constant Surplus equations for the sellers.

Efficient Matching holds: For any two skills i, j where $x_i = 0$ or $y_j = 0$, the efficient matching condition holds by definition. For any two skills i, j where $x_i > 0$ and $y_j > 0$, then $v_i^* = v_i$, $w_j^* = w_j$, and $m_{ij}^* = m_{ij}$ and the Efficient Matching condition is a direct consequence of FOC(m_{ij}).

Optimal Investments: Regarding optimal investments, just as in the proof in the main section, here the values (v_i) satisfy incentive compatibility for investments. However, it is not readily evident that the values (v_i^*) satisfy incentive compatibility because the values for unrealized skills are modified, and may be increased. We now show that for all unrealized skills $v_i \geq v_i^*$.

Since $m_{ij} x_i y_j = m_{ij}^* x_i y_j$ for any two skills i, j, the policy $\langle z, M^*, (\beta_i), (\sigma_j)\rangle$ is admissible and optimal. By the constraint qualifications, there are values $(\hat{v}_i)$, $(\hat{w}_j)$ which satisfy the FOCs for $\langle z, M^*, (\beta_i), (\sigma_j)\rangle$. From FOC($\beta_i$), we have that the marginal values are equal for all i, $\hat{v}_i - \hat{v}_{i-1} = C(i, \beta_i) - C(i-1, \beta_i) = v_i - v_{i-1}$. Likewise, for all sellers j, $\hat{w}_j - \hat{w}_{j-1} = w_j - w_{j-1}$. Thus, there is a constant t such that $\hat{v}_i + \hat{w}_j = v_i + w_j + t$ for all i, j. For any skill i such that $x_i > 0$,

$$2c = \sum_j y_j m_{ij}^* (g_{ij} - \hat{v}_i - \hat{w}_j) = \sum_j y_j m_{ij}^* (g_{ij} - v_i - w_j - t)$$
$$= \sum_j y_j m_{ij} (g_{ij} - v_i - w_j - t) = 2c - t \sum_{ij} y_j m_{ij}$$

Therefore, $t = 0$ and so $\hat{v}_i + \hat{w}_j = v_i + w_j$ for all i, j.

For any unchosen skill i,

$$\sum_j y_j m_{ij}^* \left(g_{ij} - v_i^* - w_j\right) = 2c \geq \sum_j y_j m_{ij}^* \left(g_{ij} - \hat{v}_i - \hat{w}_j\right) = \sum_j y_j m_{ij}^* \left(g_{ij} - v_i - w_j\right)$$

Therefore, we can conclude that $v_i \geq v_i^*$. This demonstrates incentive compatibility. For every skill i, $v_i \geq v_i^*$ with equality if $x_i > 0$. As (v_i) satisfied incentive compatibility and (v_i^*) differs by only lowering the value of unrealized skills, the values (v_i^*) also satisfy incentive compatibility. This establishes that for the values $(v_i^*), (w_j^*)$, no agent wishes to choose any unchosen skill and completes the proof. $\square$

Constraint Qualification

We prove the constraint qualifications.

Lemma 4. *The planner's optimization problem satisfies the Constant Rank Constraint Qualification.*

Proof. We show that for each subset of the gradients of the active inequality constraints and the equality constraints, the rank at a vicinity of the optimal point is constant (Janin [1984]).

There is an immediate linear dependency among the gradients:

$$\sum_{i \in I} \alpha \nabla flow_i^b - \sum_{j \in J} \alpha \nabla flow_j^s = 0$$

which follows from

$$\sum_{i \in I} flow_i^b - \sum_{j \in J} flow_j^s = 0$$

We will show that this is the only linear dependency, which suffices for the constant rank constraint qualification. Suppose that $\sum_n \alpha_n \nabla_n = 0$ where the summation is over all the active gradients. To simplify notation, we label the skills as $I = \{0, \dots, k\}$ and $J = \{0, \dots, l\}$. Notice first that (β_i) and (σ_j) appear only in the flow constraints:

∇	β_1	β_2	β_3	$\dots$	β_k	N	$\sigma_j, x_i,$ y_j, m_{ij}
$\nabla flow_0^b$	$-f^b(\beta_1)$	0	0	0	0	$-x_0 \sum_{j \in J} y_j m_{0j}$	$\dots$
$\nabla flow_1^b$	$f^b(\beta_1)$	$-f^b(\beta_2)$	0	0	0	$-x_1 \sum_{j \in J} y_j m_{1j}$	$\dots$
$\nabla flow_2^b$	0	$f^b(\beta_2)$	$-f^b(\beta_3)$	0	0	$-x_2 \sum_{j \in J} y_j m_{2j}$	$\dots$
$\dots$	0	0	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\nabla flow_{k-1}^b$	0	0	0	$f^b(\beta_{k-1})$	$-f^b(\beta_k)$	$-x_{k-1} \sum_{j \in J} y_j m_{k-1,j}$	$\dots$
$\nabla flow_k^b$	0	0	0	0	$f^b(\beta_k)$	$-x_k \sum_{j \in J} y_j m_{k,j}$	$\dots$

Since β_i only shows up in up in $flow_i^b$, $flow_{i-1}^b$ it must be that

$$0 = \sum_n \alpha_n \frac{\partial f_n}{\partial \beta_{i'}} = \sum_{i \in I} \alpha_i \frac{\partial flow_i^b}{\partial \beta_{i'}} = f(\beta_{i'})\alpha_{i'} - f(\beta_{i'})\alpha_{i'+1} \text{ for all } i'$$

Thus, there is an α such that $\alpha_i = \alpha$ for all the coefficients of the constraints $\nabla flow_i^b$. Similarly, there is a χ so that $\alpha_j = \chi$ for all the coefficients of the constraints $\nabla flow_j^s$. Furthermore, N only shows up in the flow constraints, so it must be that

$$-\alpha \sum_i x_i \sum_j y_j m_{ij} - \chi \sum_j y_j \sum_i x_i m_{ij} = 0$$

which implies $\chi = -\alpha$ (notice that $\sum_i x_i \sum_j y_j m_{ij} = 1/N$). Therefore, there is exactly one linear dependency

$$\sum \alpha_i \nabla flow_i^b + \sum_j \alpha_j \nabla flow_j^s = \alpha \left(\sum_i \nabla flow_i^b - \sum_j \nabla flow_j^s \right) = 0$$

Second, the coefficients on $\nabla(x_i \geq 0)$ and ∇X are all zeros. The reason is that x_i appears in the flow constraints and the constraints $x_i \geq 0$ and $X = 0$. By the previous step, in any linear dependence, the flow constraints cancel each other out, so only the constraints $x_i \geq 0$ and $X = 0$ are relevant. . Therefore, if $\sum_i \xi_i \nabla(x_i \geq 0) + \xi \nabla X = 0$, then $0 = \xi_i \frac{\partial x_i}{\partial x_i} + \xi \frac{\partial X}{\partial x_i} = \xi_i - \xi$, and so $\xi_i = \xi$ for all i. If $\xi \neq 0$, then it must be that every inequality on x is active, so $x_i = 0$ for every i, contradicting $0 = X = 1 - \sum_i x_i$, which holds in any admissible tuple. The same argument applies to the y_j. So $\xi_i = \xi = \xi_j = 0$ for all i, j.

Third, the coefficients on the m_{ij} constraints are zeros. The reason is that the variable m_{ij} appears only in the flow equations and the inequality constraints on m_{ij}. The flow constraints cancel each other out. For the m_{ij} constraints, $\nabla(1 \geq m_{ij} \geq 0) = (0, \ldots 0, \pm 1, 0 \ldots)$ and at most one of the m_{ij} constraints can be active where the only non-zero element is in the m_{ij} coordinate and therefore these gradients coefficients must be 0. $\square$

Propsition 1. *The welfare function $\mathcal{W}$ is continuous, strictly decreasing, and convex in c. Moreover, the population size N is weakly decreasing in c.*

Proof. Consider the economies $\Gamma_c = \langle F^b, F^s, I, J, C^b, C^s, G, c\rangle$ indexed by their search cost c and denote its constrained efficient welfare as $\mathcal{W}_c$. Denote an optimal allocation as x_c with associated population N_c (there may be multiple optimal allocations). Notice that by an imitation argument, $\mathcal{W}_c \geq \mathcal{W}_{c'} + 2N(c')(c' - c)$ because the planner could implement $x_{c'}$ when faced with the economy x_c. This implies that welfare is decreasing in c, as expected. Reversing c and c' gives $2N(c)(c' - c) + \mathcal{W}_{c'} \geq \mathcal{W}_c$. Taking $c' > c$. this implies that $|\mathcal{W}_c - \mathcal{W}_{c'}| \leq 2N(c)(c' - c)$. That is, when $N(c)$ is unique, it is the case that $\frac{\partial \mathcal{W}_c}{\partial c} = -2N(c)$ and otherwise the left-derivative is $\sup -2N(c)$ and the right-derivative is $\inf -2N(c)$. To see convexity of $\mathcal{W}_c$, it suffices to demonstrate that N is increasing in c. Take $c' > c$. Since $\mathcal{W}_c \geq \mathcal{W}_{c'} + 2N(c')(c' - c)$, and similarly $\mathcal{W}_{c'} \geq \mathcal{W}_c + 2N(c)(c - c')$. Adding these two equations together gives $0 > 2(N(c') - N(c))(c' - c)$ and therefore $N(c) \geq N(c')$. □

8.2 Omitted Proofs for Section 5

Assortative Matching

Theorem 2. *In equilibrium, there is PAM whenever G is supermodular, NAM whenever G is submodular, and All Skills Match whenever G is separable.*

Proof. Demonstrating PAM requires demonstrating two components, that the bounds of the matching set are weakly increasing and that the matching set is convex.

Throughout, we will use the following key fact: if G is supermodular, then so are the surpluses $[s_{ij}]$.

Increasing Upper Bounds: Fix two buyer skills $i_2 > i_1$. Suppose that $\overline{m}_{i_2} < \overline{m}_{i_1}$. Denote these as $j_2 = \overline{m}_{i_2}$ and $j_1 = \overline{m}_{i_1}$. By Efficient Matching, it must be that $s_{i_1 j_1} \geq 0 \geq s_{i_2 j_1}$. By supermodularity, then it must be that for every $j < j_1$ it is the case that $s_{i_1 j} > s_{i_2 j}$. This violates the Constant Surplus equations because

$$2c = \sum_{j \in M_{i_2}} y_j s_{i_2 j} < \sum_{j \in M_{i_2}} y_j s_{i_1 j} \leq \sum_{j \in M_{i_1}} y_j s_{i_1 j} = 2c$$

The case for lower bounds and for submodular G are analogous.

Convexity: Suppose not. That is, there is a buyer i and sellers $j_1 < j < j_2$ such that $m_{ij} < 1$, and $m_{ij_1}, m_{ij_2} > 0$. Then, it must be the case that seller j has a strictly positive surplus with a lower buyer and that buyer is present with non-zero measure. Otherwise

$$2c = \sum_{i'>i} x_i s^+_{i'j} < \sum_{i'>i} x_i s^+_{i'j_2} \leq 2c$$

with the inequality being due to the fact that $s_{i'j_2} \geq s_{ij} + s_{i'j_2} > s_{ij_2} + s_{i'j} \geq s_{ij_2}$ for every $i' > i$ due to the supermodularity of s. Therefore, there is some $i' < i$ such that $x_{i'} > 0$ and $s_{i'j} > 0$.

An analgous argument demonstrates that there is:

1. A higher buyer $i' > i$ such that $x_{i'} > 0$ and $s_{i'j} > 0$.
2. A lower seller $j' < j$ such that $y_{j'} > 0$ and $s_{ij'} > 0$.
3. A higher seller $j' > j$ such that $y_{j'} > 0$ and $s_{ij'} > 0$.

Let $\underline{j} = arg\max_{j' \leq j} s_{ij'}$ and likewise $\overline{j} = arg\max_{j' \geq j} s_{ij'}$. Similarly, let $\underline{i} = arg\max_{i' \leq i} s_{i'j}$ and likewise $\overline{i} = arg\max_{i' \geq i} s_{i'j}$. See below for an illustration of the matching matrix.

	$\dots$	$\underline{j}$	$\dots$	j	$\dots$	$\overline{j}$	$\dots$
$\underline{i}$				1			
$\dots$							
i		1		$m_{ij} < 1$		1	
$\dots$							
$\overline{i}$				1			
$\dots$							

Define $y = y_j$, $\underline{y} = \sum_{j'<j} y_{j'}$ and $\overline{y} = \sum_{j'>j} y_{j'}$. Similarly, $x = x_i$, $\underline{x} = \sum_{i'<i} x_{i'}$, and $\overline{x} = \sum_{i'>i} x_{i'}$. Notice that $\overline{x}, \underline{x}, \overline{y}, \underline{y} > 0$ as shown above.

By the supermodularity of s, for any $i' > i$, it is the case that $s_{i'\overline{j}} + s_{ij} > s_{i'j} + s_{i\overline{j}}$ and since $s_{ij} \leq 0$, it follows that $s_{i'\overline{j}} > s_{i'j} + s_{i\overline{j}}$. Thus,

$$2c \geq \sum_{i' \geq i} x_{i'} s_{i'\overline{j}} > \sum_{i' \geq i} x_{i'}(s_{i'j} + s_{i\overline{j}}) = \left(\sum_{i' \geq i} x_{i'} s_{i'j}\right) + (x + \overline{x}) s_{i\overline{j}} \tag{13}$$

The strict inequality use the fact that $x_{i'} > 0$ for some $i' > i$.

Next, notice that $s_{\underline{i}j} \geq s_{i'j}$ for all $i' < i$. Therefore,

$$\underline{x}s_{\underline{i}j} = \sum_{i'<i} x_{i'}s_{\underline{i}j} \geq \sum_{i'<i} x_{i'}s_{i'j} \tag{14}$$

Adding equations (13) and (14) gives

$$2c + \underline{x}s_{\underline{i}j} > \sum_{i'} x_{i'}s_{i'j} + (x + \overline{x})s_{i\overline{j}}$$

And therefore,

$$\underline{x}s_{\underline{i}j} > (x + \overline{x})s_{i\overline{j}}$$

Similarly, it can be observed that:

$$\begin{aligned} s_{\overline{i}j'} &> s_{ij'} + s_{\overline{i}j} \text{ for all } j > j' \\ s_{\underline{i}j'} &> s_{ij'} + s_{\underline{i}j} \text{ for all } j' > j \\ s_{i'\underline{j}} &> s_{i'j} + s_{i\underline{j}} \text{ for all } j' < j \end{aligned}$$

Repeating the same arguments as above:

$$\begin{aligned} \overline{y}s_{i\underline{j}} &> (\underline{y} + y)s_{\overline{i}j} \\ \overline{y}s_{i\overline{j}} &> (y + \underline{y})s_{\underline{i}j} \\ \overline{x}s_{\overline{i}j} &> (\underline{x} + x)s_{i\underline{j}} \end{aligned}$$

As shown earlier, all of the surpluses, $s_{\underline{i}j}, s_{\overline{i}j}, s_{i\underline{j}}, s_{i\overline{j}}$ are positive. Taking the product of the above four inequalities and dividing by the surpluses yields:

$$\underline{x}\overline{x}\underline{y}\overline{y} > (\underline{x} + x)(\overline{x} + x)(\underline{y} + y)(\overline{y} + y)$$

which is a contradiction due to the strict inequality.

Separability Implies All Skills Match: By Lemma 2, it is the case that for any two sellers, $w_{j'} - w_j = g_{j'} - g_j$. Therefore, the surplus function is constant $s_{ij'} = g_i + g_{j'} - v_i - w_{j'} = g_i + g_j - v_i - w_j$ and by the Constant Surplus equations, it must be that $s_{ij} = 2c$ for all i, j. So, every pair of agents accept their match. $\square$

Proofs for Subsection 5.3 (Marriage Market)

Proposition 3. *If G is supermodular, every equilibrium is symmetric, that is, $x = y$ and $\Delta v = \Delta w$ and $M = M^T$.*

Proof. Recall that $x = x_1$ and $y = y_1$. The case where x or y is non-interior is trivial. For example, suppose $x = 1$, that is, all buyers invest. Then, by Lemma 2, it is the case that $\Delta w = g_{11} - g_{10}$ and $g_{11} - g_{10} \geq \Delta v \geq g_{10} - g_{00}$. So, the sellers have a weakly greater incentive to invest than buyers and it must be that all sellers invest $y = 1$. Thus, again by Lemma 2, $\Delta v = g_{11} - g_{10} = \Delta w$. Finally, $2c = g_{11} - v_1 - w_1 = g_{10} - v_1 - w_0 = g_{01} - v_0 - w_1 = g_{01} - g_{11} + g_{10} - v_0 - w_0 < g_{00} - v_0 - w_0$ where the last inequality is by supermodularity, so all agents match and M is symmetric. The cases where $x = 0, y = 1$, or $y = 0$ are analogous.

For the interior case, let $s^+_{ij} = \max\{0, g_{ij} - v_i - w_j\}$. The Constant Surplus equations are

$$ys^+_{11} + (1-y)s^+_{10} = 2c$$

$$ys^+_{01} + (1-y)s^+_{00} = 2c$$

$$xs^+_{11} + (1-x)s^+_{01} = 2c$$

$$xs^+_{10} + (1-x)s^+_{00} = 2c$$

$M = M^T$**:** Suppose not and WLOG that $m_{10} > m_{01}$. By Efficient Matching, $v_1 + w_0 \leq g_{10} = g_{01} \leq v_0 + w_1 \Rightarrow \Delta v \leq \Delta w$. By the Inflow=Outflow equations,

$$F(\Delta w) = Ny(m_{11}x + m_{01}(1-x)) \geq Nx(m_{11}y + m_{10}(1-y)) = F(\Delta v)$$

$$\Rightarrow ym_{01}(1-x) \geq xm_{10}(1-y) \Rightarrow y > x \tag{15}$$

However, since $m_{01} < 1$, by Efficient Matching, $s^+_{01} = 0$, and two of the Constant Surplus equations are,

$$ys^+_{11} + (1-y)s^+_{10} = 2c$$

$$xs^+_{11} = 2c$$

which implies that $x \geq y$, a contradiction.

$x = y$: Suppose $s_{10} > s_{01}$. Then, $g_{10} - v_1 - w_0 > g_{01} - v_0 - w_1$ which implies $\Delta w > \Delta v$. By Equation (15), it must be that $y > x$. But, then the two Constant

Surplus equations are:

$$ys_{11}^{+} + (1-y)s_{10}^{+} = 2c$$

$$xs_{11}^{+} + (1-x)s_{01}^{+} = 2c$$

By Lemma 2, $s_{11} > s_{10}$. This is a contradiction because $ys_{11}^{+} + (1-y)s_{10}^{+} > xs_{11}^{+} + (1-x)s_{01}^{+}$ as the left hand side is a convex combination of larger numbers which puts higher weight on the largest number.

$\Delta v = \Delta w$: Since $M = M^T$and $x = y$, the outflow of skilled buyers equals the outflow of skilled sellers and therefore their inflows must also be the same $F(\Delta v) = F(\Delta w)$. Since F has non-zero density, $\Delta v = \Delta w$. □

Proposition 4. *If G is submodular and F is centered, then for sufficiently small c, there is a unique equilibrium: it is symmetric and satisfies $x = y = 1/2$, $N = 2$, $\Delta v = \Delta w$, and $M = M^{NAM}$.*

The proof relies on the following two Lemmas.

Lemma 5. *In any two-skill economy, every equilibrium has $N \leq 2$.*

Proof. G can be separable, submodular, or supermodular. When G is separable, we have already proven that $N = 1$. We now show the submodular case (the supermodular case follows a similar argument).

From Theorem 2, in any equilibrium, $m_{10} = m_{01} = 1$. Thus, 3 types of matching matrices can occur.

All 1's, M $= \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$**:** The total outflow of buyers is $1 = N(x + (1-x)) = N$.

Three 1's, WLOG $M = \left[\begin{smallmatrix} 1 & 1 \\ 1 & m_{11} \end{smallmatrix}\right]$ where $m_{11} < 1$**:** The skilled buyer and unskilled seller Constant Surplus equations are $(1-y)s_{10} = 2c = xs_{10} + (1-x)s_{00}$. Therefore, $1 - y \geq x$. The outflow of unskilled buyers is $N(1-x) \leq 1$ and unskilled sellers is $N(1-y) \leq 1$. Summing these $2 \geq N(1 - x + (1-y)) \geq N(1 - x + x) = N$. This argument can be repeated for any other matching matrix with exactly three 1's.

Two 1's, WLOG $M = \left[\begin{smallmatrix} m_{00} & 1 \\ 1 & m_{11} \end{smallmatrix}\right]$ where $m_{00}, m_{11} < 1$: The skilled buyer and unskilled seller Constant Surplus equations are, $(1-y)s_{10} = 2c = xs_{10} \Rightarrow 1 - y = x$. The total outflow of buyers is $1 = N(xym_{11} + x(1-y) + (1-x)y + (1-x)(1-y)m_{00}) \geq N(x(1-y) + (1-x)y) = N(x^2 + (1-x)^2) \geq N/2$.

□

Lemma 6. *If F has a large support, there exists $\delta > 0$ such that for any c the equilibrium state is uniformly bounded: $1 - \delta > x, y > \delta$.*

Proof. By Lemma 2, $\Delta v, \Delta w$ are bounded by the marginal productivities, $g_{11} - g_{10}$ and $g_{10} - g_{00}$. Therefore, the inflow of skilled buyers is between $F(g_{11} - g_{10})$ and $F(g_{10} - g_{00})$. Since F has large support, the inflows are uniformly bounded, and by the Inflow=Outflow equations and the fact that $N \leq 2$, the stocks are uniformly bounded as well. □

Proof of Proposition 4:

Proof.
Step 1: Show that $M = M^{NAM}$ by using Lemma 5.

By Lemma 6, the state x, y are uniformly bounded away from $[0, 1]$. That is, there exists some $\delta > 0$ such that $\delta < x, y < 1 - \delta$. From Theorem 2, in any equilibrium, $m_{10} = m_{01} = 1$. Therefore, the Constant Surplus equations are:

$$s_{10} = g_{10} - v_1 - w_0 < \frac{2c}{\delta}$$
$$s_{01} = g_{01} - v_0 - w_1 < \frac{2c}{\delta}$$

As g is submodular, we can take a c small enough so that $2c/\delta < -\Delta g/2$. Therefore, $v_1 + w_1 + v_0 + w_0 > g_{10} + g_{01} + \Delta g = g_{11} + g_{00}$ it must be that at least one of s_{00} or s_{11} is negative. Let us assume that $s_{11} < 0$ and $s_{00} \geq 0$, then $m_{11} = 0$ and the CS equations are

$$(1 - y)s_{10} = 2c$$
$$ys_{01} + (1 - y)s_{00} = 2c$$
$$(1 - x)s_{01} = 2c$$
$$xs_{10} + (1 - x)s_{00} = 2c$$

implying marginal values, $\Delta w = g_{10} - g_{00} - \frac{2cx}{(1-x)(1-y)}$ and $\Delta v = g_{10} - g_{00} - \frac{2cy}{(1-x)(1-y)}$. Since x and y are bounded away from 1 and F is centered, $F(g_{10} - g_{00}) > 1/2$, then for c sufficiently small, $p = F(\Delta v) > 1/2$ and $q = F(\Delta w) > 1/2$. The Inflow=Outflow equations are:

$$\frac{x(1-y)}{x(1-y)+(1-x)y+(1-x)(1-y)m_{00}} = p$$
$$\frac{(1-x)y}{x(1-y)+(1-x)y+(1-x)(1-y)m_{00}} = q$$

imply $p+q \leq 1$, a contradiction.

If $s_{11} \geq 0$ and $s_{00} < 0$, then $m_{00} = 0$, and we apply the same argument,

$$(1-y)s_{10} + ys_{11} = 2c$$

$$ys_{01} = 2c$$

$$(1-x)s_{01} + xs_{11} = 2c$$

$$xs_{10} = 2c$$

Implying the marginal values, $\Delta w = g_{11} - g_{10} + \frac{2c(1-x)}{xy}$ and $\Delta v = g_{11} - g_{01} + \frac{2c(1-y)}{xy}$. Since F is centers, $p = F(\Delta v) < 1/2$ and $q = F(\Delta w) < 1/2$, for sufficiently small c. The Inflow=Outflow equations are:

$$\frac{(1-x)y}{xym_{11}+x(1-y)+(1-x)y} = 1-p$$
$$\frac{(1-y)x}{xym_{11}+x(1-y)+(1-x)y} = 1-q$$

implying that $p+q > 1$, a contradiction.

Therefore, it must be that $s_{11} < 0$ and $s_{00} < 0$. Therefore, by Efficient Matching, it is the case that $m_{00} = m_{11} = 0$.

Step 2: Show that $x = 1-y$ and the fundamental equation (16) both hold.

Since $m_{11} = m_{00} = 0$ and $m_{01} = m_{10} = 1$, two of the constant Surplus equations are $(1-y)s_{10} = 2c = xs_{10}$ and thus $1-y = x$. Therefore, the Constant Surplus equations reduce to

$$x(g_{10} - v_1 - w_0) = 2c$$

$$(1-x)(g_{01} - v_0 - w_1) = 2c$$

Dividing and taking the difference gives $\Delta v - \Delta w = \frac{2c}{1-x} - \frac{2c}{x}$. The Inflow=Outflow equations require that $F(\Delta v) = Nx(1-y)$, $F(\Delta w) = N(1-x)y$, and $1 = Nx(1-y) + N(1-x)y$, which together with $x = 1-y$ yields $F(\Delta v) = \frac{x^2}{x^2+(1-x)^2}$ and $F(\Delta w) = \frac{(1-x)^2}{x^2+(1-x)^2}$. The two conditions on the difference between the marginal values imply,

$$F^{-1}\left(\frac{x^2}{x^2+(1-x)^2}\right) - F^{-1}\left(\frac{(1-x)^2}{x^2+(1-x)^2}\right) = \frac{2c}{1-x} - \frac{2c}{x} \tag{16}$$

Step 3: Show that there is a unique solution to Equation (16).

The value $x = 1/2$ solves Equation (16). To see that there is no other, we take the derivatives of both sides of 16 and show that they are ranked for $c$small enough:

$$\frac{2x(1-x)}{(x^2+(1-x)^2)^2 f(F^{-1}(z))} + \frac{2x(1-x)}{(x^2+(1-x)^2)^2 f(F^{-1}(1-z))} > 2c\left(\frac{1}{x^2} + \frac{1}{(1-x)^2}\right)$$

where $z = \frac{x^2}{x^2+(1-x)^2}$. To see the ranking, since x is uniformly bounded, there is some ϵ so that the LHS is strictly greater than ϵ for all $x \in [\delta, 1-\delta]$ and then there is some c small enough so that the RHS is less than ϵ for all $x \in [\delta, 1-\delta]$. Recall that in every equilibrium, $x \in [\delta, 1-\delta]$ and therefore, there is at most one equilibrium.

Step 4: $x = y = 1/2$, $N = 2$, $M = M^{NAM}$and $\Delta v = \Delta w = \mu$ is an equilibrium.

From the Second Welfare Theorem, we know that an equilibrium exists, and since there is a unique equilibrium candidate (from the conditions above), we know that it is indeed an equilibrium. In particular, it is straightforward to verify that the above variables with the following values constitute an equilibrium: $v_L = w_L = \frac{g_{10} - 4c - med\,F}{2}$ and $v_H = w_H = \frac{g_{10} - 4c + med\,F}{2}$. □

Lemma 3. *For c sufficiently large, in every equilibrium $M = M^{All}$.*

Proof. WLOG suppose $g_{ij} \geq 0, \forall i, j$. Define $\overline{g} = \max g_{ij}$ and let $c = 2\overline{g}$. The buyers' Constant Surplus equations are $\sum_j y_j s_{ij}^+ = 2c \geq 4\overline{g}$. Thus, for each i there is some j such that $4\overline{g} \leq s_{ij}^+ = g_{ij} - v_i - w_j$. Therefore, $v_i + w_j < -2\overline{g}$. By Lemma (2), for any other j', it is the case that $w_{j'} - w_j \leq \max_i g_{ij'} - g_{ij} \leq \overline{g}$. Thus, for any other j', $v_i + w_{j'} = v_i + w_j + (w_{j'} - w_j) \leq v_i + w_j + \overline{g} < -2\overline{g} + \overline{g} < 0 \leq g_{ij}$. So, by Efficient Matching, all agents match with probability 1. □

Proposition 5. *If G is submodular and $|\Delta g|\, f(\beta^{sym}) > 1$, then for sufficiently large c, the efficient equilibrium is asymmetric.*

Proof.

Step 1. Lemma 3 shows that for large c, in every equilibrium All Skills match.

Step 2. The efficient equilibrium is asymmetric.

If all skills match and the buyer/seller investment thresholds are β_1, σ_1, then social welfare is

$$W = xyg_{11} + x(1-y)g_{10} + (1-x)yg_{01} + (1-y)(1-x)g_{00} - \int_0^{\beta_1} \beta f(\beta)d\beta - \int_0^{\sigma_1} \sigma f(\sigma)d\sigma - 2c$$

where $x = F(\beta_1)$ and $y = F(\sigma_1)$. We will show if $f(\beta^{sym})|\Delta g| > 1$, then the solution is not symmetric.

FOC:

$$\begin{aligned} W_{\beta_1} &= f(\beta_1)\left(yg_{11} + (1-y)g_{10} - yg_{01} - (1-y)g_{00}\right) - \beta_1 f(\beta_1) = 0 \\ W_{\sigma_1} &= f(\sigma_1)\left(xg_{11} - xg_{10} + (1-x)g_{01} - (1-x)g_{00}\right) - \sigma_1 f(\sigma_1) = 0 \end{aligned}$$

Implying, $\beta_1 = y\left(g_{11} - g_{01}\right) + (1-y)\left(g_{10} - g_{00}\right)$ and $\sigma_1 = x\left(g_{11} - g_{10}\right) + (1-x)\left(g_{01} - g_{00}\right)$.

SOC:

$$\begin{aligned} W_{\beta_1\beta_1} &= f'(\beta_1)\left(y\left(g_{11} - g_{01}\right) + (1-y)\left(g_{10} - g_{00}\right)\right) - f(\beta_1) - \beta_1 f'(\beta_1) = -f(\beta_1) \\ W_{\sigma_1\sigma_1} &= f'(\sigma_1)\left(x\left(g_{11} - g_{10}\right) + (1-x)\left(g_{01} - g_{00}\right)\right) - f(\sigma_1) - \sigma_1 f'(\sigma_1) = -f(\sigma_1) \\ W_{\beta_1\sigma_1} &= f(\beta_1)f(\sigma_1)\left(\Delta g\right) \end{aligned}$$

If (β_1, σ_1) satisfy the FOC, then it is a saddle point whenever

$$W_{\beta_1\beta_1}W_{\sigma_1\sigma_1} - \left(W_{\beta_1\sigma_1}\right)^2 = f(\beta_1)f(\sigma_1) - \left(f(\beta_1)f(\sigma_1)\right)^2\left(\Delta g\right)^2 < 0$$

or in other words, when

$$f(\beta_1)f(\sigma_1)\left(\Delta g\right)^2 > 1$$

Therefore, the symmetric solution is a saddle point if $f(\beta^{sym})|\Delta g| > 1$, which completes the proof. □

Proposition 6. *Let G be supermodular. For sufficiently small c, every equilibrium satisfies the following conditions: $M = M^{PAM}$, $x = y$, $N = \frac{1}{x^2+(1-x)^2}$, and*

$$\overbrace{F(\Delta v) = F\left(\frac{g_{11}}{2} - \frac{c}{x} - \left(\frac{g_{00}}{2} - \frac{c}{1-x}\right)\right)}^{inflow} = \overbrace{\frac{x^2}{x^2+(1-x)^2}}^{outflow} \tag{17}$$

Furthermore, these conditions are also sufficient: if z satisfies them, then there are values (v_i), (w_j) such that $\langle z, M^{PAM}, (v_i), (w_j)\rangle$ is an equilibrium.

Proof.
Step 1: We show that if c is sufficiently small, $M = M^{PAM}$. From Theorem 2, we know that $m_{11} = m_{00} = 1$.

By Lemma 6, the state is uniformly bounded, $1-\delta > x > \delta > 0$. The Constant Surplus equations are:

$$xs_{11} + (1-x)s_{10}^{+} = 2c \Rightarrow s_{11} \leq \frac{2c}{x}$$
$$(1-x)s_{00} + xs_{01}^{+} = 2c \Rightarrow s_{00} \leq \frac{2c}{1-x}$$

Therefore, for all $c < c' = \delta \times |\Delta g|/2$,

$$s_{11} = g_{11} - v_1 - w_1 < \frac{2c}{\delta} < \Delta g/2$$
$$s_{00} = g_{00} - v_0 - w_0 < \frac{2c}{\delta} < \Delta g/2.$$

Summing these two inequalities, $v_1+w_0+v_0+w_1 > g_{11}+g_{00}-\Delta g = 2g_{10}$. Therefore, it must be that either $s_{10} = g_{10} - v_1 - w_0 < 0$ or $s_{01} = g_{01} - v_0 - w_1 < 0$, so at least one of m_{10}, m_{01} equals 0. From Proposition 3, we know that $m_{01} = m_{10}$, so it must be that $m_{10} = m_{01} = 0$.

Step 2: We derive necessary and sufficient conditions for the PAM equilibrium.

Since $m_{10} = m_{01} = 0$, the Constant Surplus equations imply $ys_{11} = 2c = xs_{11}$, and thus $x = y$. These equations reduce to $s_{11} = \frac{2c}{x}$ and $s_{00} = \frac{2c}{1-x}$. Subtracting implies that $\Delta w + \Delta v = g_{11} - \frac{2c}{x} - \left(g_{00} - \frac{2c}{1-x}\right)$. Furthermore, to maintain a steady state, skilled buyers and skilled sellers must enter at the same rate (because skilled buyers and sellers exit at the same rate), and so $\Delta w = \Delta v = \frac{g_{11}}{2} - \frac{c}{x} - \left(\frac{g_{00}}{2} - \frac{c}{1-x}\right)$. The Inflow=Outflow equations require that $F(\Delta v) = Nx^2$ and $1 = Nx^2 + N(1-x)^2$ which together impose a further restriction

$$\overbrace{F(\Delta v) = F\left(\frac{g_{11}}{2} - \frac{c}{x} - \left(\frac{g_{00}}{2} - \frac{c}{1-x}\right)\right)}^{inflow} = \overbrace{\frac{x^2}{x^2 + (1-x)^2}}^{outflow} \tag{18}$$

The Efficient Matching condition is:[23]

$$\Delta g x\,(1-x) - 2c \geq 0 \tag{19}$$

which holds if c is sufficiently small (in particular, if $c < \Delta g\delta(1-\delta)/2$).

Step 3: The conditions are sufficient. Given an x satisfying conditions (18) and $c$sufficiently small, it is straightforward to verify that $v_1 = w_1 = \frac{g_{11}}{2} - \frac{c}{x}$, $v_0 = w_0 = \frac{g_{00}}{2} - \frac{c}{1-x}$ and $M = M^{PAM}$ are an equilibrium. □

Proposition 7. *For sufficiently large c, every equilibrium satisfies the following conditions: $M = M^{All}$, $N = 1$, and*

$$x = F\left(\Delta v\right) = F\left(y(g_{11} - g_{10}) + (1-y)(g_{01} - g_{00})\right) \tag{20}$$

$$y = F\left(\Delta w\right) = F\left(x(g_{11} - g_{01}) + (1-x)(g_{10} - g_{00})\right) \tag{21}$$

Furthermore, these conditions are also sufficient: if x, y satisfies (20) and (21), then there are values such that $\langle z, M^{All}, v, w\rangle$ is an equilibrium.

Proof.

Step 1: By Lemma 3, for c sufficiently large, All Skills Match, $M = M^{All}$.

Step 2: The above conditions are necessary. If all skills match, then Lemma 2 implies that the marginal values equal the marginal productivity,

$$\begin{aligned} x = F(\Delta v) = F\left(y(g_{11} - g_{10}) + (1-y)(g_{01} - g_{00})\right) \\ y = F(\Delta w) = F\left(x(g_{11} - g_{01}) + (1-x)(g_{10} - g_{00})\right) \end{aligned} \tag{22}$$

[23]To explain, since $\Delta v = \Delta w$, it follows that $v_1 + w_0 = v_0 + w_1$. Thus, $v_1 + w_0 \geq g_{10} \Leftrightarrow v_0 + w_1 \geq g_{10} \Leftrightarrow v_1 + w_0 + v_0 + w_1 \geq 2g_{10}$. Substituting in the Constant Surplus equations, we get condition (19).

To satisfy the Efficient Matching conditions, the Constant Surplus equations imply

$$\begin{aligned} s_{11} \geq 0 &\iff -\Delta g(1-x)(1-y) \leq 2c \\ s_{01} \geq 0 &\iff \Delta g x(1-y) \leq 2c \\ s_{10} \geq 0 &\iff \Delta g(1-x)y \leq 2c \\ s_{00} \geq 0 &\iff -\Delta g x y \leq 2c \end{aligned} \tag{23}$$

Sufficiency: Take x and y satisfying the above conditions, we set $\Delta v = F^{-1}(x)$ and $\Delta w = F^{-1}(y)$, $y(g_{11} - v_1 - w_1) + (1-y)(g_{10} - v_1 - w_0) = 2c$, and $w_0 = 0$. □

8.3 Omitted Proofs for Section 6

Proofs of Corollaries 2 and 5:

Corollary 2. *In a model with outside options, the constrained efficient outcome is an equilibrium.*

Corollary 5. *The constrained efficient allocation is an equilibrium if and only if*

$$\alpha = \frac{B^*c^b}{B^*c^b + S^*c^s} = \frac{\partial\mu(B^*, S^*)/\partial B}{\mu(B^*, S^*)/B^*}$$

where B^*, S^* are the constrained efficient stock.

Proof. To simplify, we focus on the case where the state is interior and the proof repeats that argument with the appropriate modifications. The same could be done for the boundary case as well. Recall that $\mu(B, S)$ is the number of meetings in every period. The original planner's problem 5 is modified because the agents have an outside option and there is a general meeting function, and so the measure of buyers B need not equal the measure of sellers S. The planner now chooses the state $z = (B, S, (x_i), (y_j))$ instead of $z = (N, (x_i), (y_j))$, the investment thresholds, and the matching rule to maximize

$$\begin{aligned} \mathcal{W} = \mu(B, S) \sum_{i \in I} \sum_{j \in J} x_i y_j m_{ij} g_{ij} - Bc^b - Sc^s - \sum_{i \in I} \int_{\beta_{i+1}}^{\beta_i} C^b(i, \beta) f^b(\beta) d\beta \\ - \sum_{j \in J} \int_{\sigma_{j+1}}^{\sigma_j} C^s(j, \sigma) f^s(\sigma) d\sigma + \int_{\beta_0}^{\infty} u^b f^b(\beta) d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma) d\sigma \end{aligned}$$

subject to the steady state conditions,

$$
\begin{aligned}
flow_i &= \int_{\beta_{i+1}}^{\beta_i} f^b(\beta)d\beta - x_i\mu(B,S)\sum_{j\in J} y_j m_{ij} = 0, \forall i \\
flow_j &= \int_{\sigma_{j+1}}^{\sigma_j} f^s(\sigma)d\sigma - y_j\mu(B,S)\sum_{i\in I} x_i m_{ij} = 0, \forall j \\
& B, S \geq 0 \\
& x_i \geq 0, \forall i \\
& y_j \geq 0, \forall j \\
X &= 1 - \sum_{i\in I} x_i = 0 \\
Y &= 1 - \sum_{j\in J} y_j = 0 \\
& 1 \geq m_{ij} \geq 0, \forall i, j \\
F^b(\beta_{|I|}) &= F^s(\sigma_{|J|}) = 0
\end{aligned}
$$

Notice that taking weighted sums of the flow conditions implies that $F^b(\beta_0) = F^s(\sigma_0)$. The planner's problem is modified in three ways: i) agents can take an outside option which is included in the objective function and the conditions $F(\beta_0) = 1$ and $F(\sigma_0) = 1$ are removed; ii) the measure of buyers B and sellers S may differ and since we assumed that the are gains to trade, the conditions $B, S \geq 0$ will not bind at the efficient solution; iii) the Inflow=Outflow equations are modified because the outflow of buyers and sellers is

$$
\begin{aligned}
(Bx_i)\left(\frac{\mu(B,S)}{B}\right)\sum_{j\in J} y_j m_{ij} &= x_i\mu(B,S)\sum_{j\in J} y_j m_{ij},\ \forall i \\
(Sy_j)\left(\frac{\mu(B,S)}{S}\right)\sum_{i\in I} x_i m_{ij} &= y_j\mu(B,S)\sum_{i\in I} x_i m_{ij},\ \forall j
\end{aligned}
$$

The KKT conditions regularity conditions continue to hold, by the same arguments as in Theorem 1 (because the linear dependencies of the gradients do not change).

The Lagrangian is

$$
\begin{aligned}
\mathcal{L} =& \mu(B,S)\sum_I\sum_j x_i y_j m_{ij} g_{ij} - Bc^b - Sc^s - \sum_I \int_{\beta_{i+1}}^{\beta_i} c(i,\beta) f^b(\beta) d\beta \\
& - \sum_J \int_{\sigma_{j+1}}^{\sigma_j} c(j,\sigma) f^s(\sigma) d\sigma + \int_{\beta_0}^{\infty} u^b f^b(\beta) d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma) d\sigma \\
& + \sum_{i\in I} v_i flow_i + \sum_{j\in J} w_j flow_j + \sum_i \phi_i x_i + \sum_j \psi_j y_j + \gamma X + \lambda Y \\
& + \sum_{i\in I}\sum_{j\in J} (\eta_{ij} m_{ij} + \hat{\eta}_{ij}(1-m_{ij}))
\end{aligned}
$$

FOC(B): $$(\partial\mu/\partial B)\left(\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij}\left(g_{ij} - v_i - w_j\right)\right) - c^b = 0$$

$$\Rightarrow \sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} s_{ij} = \frac{c^b}{\partial\mu/\partial B}$$

FOC(x$_\mathbf{i}$): $$\mu\sum_{j\in J} y_j m_{ij} g_{ij} - v_i\mu\sum_{j\in J} y_j m_{ij} - \mu\sum_{j\in J} w_j y_j m_{ij} - \gamma - \phi_i = 0$$

$$\sum_{j\in J} y_j m_{ij} s_{ij} = \frac{\gamma + \phi_i}{\mu}$$

and $x_i\phi_i = 0$.

Thus, substituting FOC(x_i) into FOC(B), the second into the first, we get $\frac{\gamma}{\mu} = \frac{c^b}{\partial\mu/\partial B}$ (because $\sum_{i\in I} x_i = 1$ and $x_i\phi_i = 0$). Thus

$$\sum_J y_j m_{ij} s_{ij} = \frac{c^b}{\partial\mu/\partial B} + \frac{\phi_i}{\mu}$$

and if $\phi_i = 0$, then

$$\sum_J y_j m_{ij} s_{ij} = \frac{c^b}{\partial\mu/\partial B} \tag{24}$$

We now do the same for the sellers.

FOC(S): $$\sum_{j\in J}\sum_{i\in I} x_i y_j m_{ij} s_{ij} = \frac{c^s}{\partial\mu/\partial S}$$

FOC($\mathbf{y_j}$): $$\mu \sum_I x_i m_{ij} g_{ij} - w_j \mu \sum_I x_i m_{ij} - \sum_I v_i \mu x_i m_{ij} - \eta - \psi_j = 0$$

$$\sum_I x_i m_{ij} s_{ij} = \frac{\lambda + \psi_j}{\mu}$$

and $\psi_j y_j = 0$. Thus,

$$\sum_I x_i m_{ij} s_{ij} = \frac{c^s}{\partial\mu/\partial S} \tag{25}$$

Decentralizing the optimal allocation: we show that the shadow values v_i, w_j together with the matching matrix M and state z constitute an equilibrium, provided that the bargaining weight is $\alpha = \frac{\partial\mu/\partial B}{\mu/B}$. To see why, substitute $\partial\mu/\partial B = \alpha\,(\mu/B)$ into condition (24)

$$\sum_{j\in J} y_j m_{ij} s_{ij} = \frac{c^b}{\alpha\,(\mu/B)}, \forall i$$

which is the Constant Surplus equation for skill i.

For sellers, since μ is homogeneous of degree 1,[24] $\frac{\partial\mu/\partial S}{\mu/S} = 1 - \frac{\partial\mu/\partial B}{\mu/B}$, and thus, $1-\alpha = \frac{\partial\mu/\partial S}{\mu/S}$. Substituting into equation (25) gives the sellers' Constant Surplus equations:

$$\sum_{i\in I} x_i m_{ij} s_{ij} = \frac{c^s}{(1-\alpha)\,(\mu/S)}, \forall j$$

The FOC(β_0) condition is precisely the equilibrium entry condition, $v_0 - C(0,\beta_0) = u^b$, and so the shadow value v_0 and threshold β_0 satisfy the equilibrium entry condition. Likewise, the seller's entry condition holds as well. The proofs that the Efficient Matching conditions and individual optimal investments hold are the same as in Theorem 1.

Furthermore, by FOC(B) and FOC(S), we have that $c^s\,(\partial\mu/\partial B) = c^b\,(\partial\mu/\partial S)$. By homogeneity of degree 1,

$$B\,(\partial\mu/\partial B) + S(\partial\mu/\partial S) = \mu \Rightarrow c^b\,[B\,(\partial\mu/\partial B) + S(\partial\mu/\partial S)] = c^b\mu$$

Substituting in gives:

$$Bc^b\,(\partial\mu/\partial B) + Sc^s\,(\partial\mu/\partial B) = c^b\mu \Rightarrow \frac{\partial\mu/\partial B}{\mu} = \frac{c^b}{Bc^b + Sc^s}$$

[24]Homogeneity of degree 1 implies $B\,(\partial\mu/\partial B) + S(\partial\mu/\partial S) = \mu \iff (\partial\mu/\partial B)\,/\,(\mu/B) + (\partial\mu/\partial S)/\,(\mu/S) = 1$.

Therefore, the buyers' bargaining weight $\alpha = \frac{\partial\mu/\partial B}{\mu/B} = \frac{Bc^b}{Bc^b+Sc^s}$ and so the seller's bargaining weight is $1-\alpha = \frac{Sc^s}{Bc^b+Sc^s}$.

$\Leftarrow$ Recall that, in an equilibrium, the Constant Surplus equations imply Equation (12) $\frac{B}{S} = \frac{\alpha}{1-\alpha}\cdot\frac{c^s}{c^b}$. Therefore, if the constrained efficient solution is an equilibrium, it must be that $\frac{c^s}{c^b} = \frac{(1-\alpha)B}{\alpha S}$ and therefore, it must be that $\alpha = \frac{Bc^b}{Bc^b+Sc^s}$. $\square$

Corollary 4. *For any bargaining weight α, and search costs c^b and c^s, an equilibrium exists. Moreover, in every equilibrium, there is PAM (NAM) whenever G is supermodular (submodular).*

Proof. Assume WLOG that $\frac{\alpha}{1-\alpha}\cdot\frac{c^s}{c^b} > 1$. We set $c = \frac{c^s}{2(1-\alpha)}$ and define the auxiliary problem:

$$\mathcal{W} = N\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} g_{ij} - 2Nc - \sum_{i\in I}\int_{\beta_{i+1}}^{\beta_i} C^b(i,\beta)f^b(\beta)d\beta - \sum_{j\in J}\int_{\sigma_{j+1}}^{\sigma_j} C^s(j,\sigma)f^s(\sigma)d\sigma$$

subject to

$$\begin{aligned}
flow_i = \int_{\beta_{i+1}}^{\beta_i} f^b(\beta)d\beta - x_i N\sum_{j\in J} y_j m_{ij} &= 0, \forall i\\
flow_j = \int_{\sigma_{j+1}}^{\sigma_j} f^s(\sigma)d\sigma - y_j N\sum_{i\in I} x_i m_{ij} &= 0, \forall j\\
N &\geq 0\\
x_i &\geq 0, \forall i\\
y_j &\geq 0, \forall j\\
X = 1 - \sum_{i\in I} x_i &= 0\\
Y = 1 - \sum_{j\in J} y_j &= 0\\
1 \geq m_{ij} &\geq 0, \forall i, j\\
F^b(\beta_{|I|}) = F^s(\sigma_{|J|}) &= 0
\end{aligned}$$

The above problem has a solution. Let v_i and w_j denote the shadow values of the flow constraints, $z^* = \langle N^*, (x_i^*), (y_j^*)\rangle$ denote the optimal state variable, and M^* denotes the optimal matching matrix. Set $\hat{S} = N^*$ and $\hat{B} = N^*\left(\frac{\alpha}{1-\alpha}\cdot\frac{c^s}{c^b}\right)$, and $z = \langle\hat{B}, \hat{S}, (x_i^*), (y_j^*)\rangle$. We argue that $\langle z, M^*, (v_i), (w_j)\rangle$ constitutes an equilibrium.

Notice that the steady state flow equations hold by construction. The Constant Surplus equations are

$$\sum_{j\in J} y_j m_{ij} s_{ij} = \frac{c^b B}{\alpha \min(B,S)}, \forall i \qquad (26)$$
$$\sum_{i\in I} x_i m_{ij} s_{ij} = \frac{c^s S}{(1-\alpha)\min(B,S)}, \forall j$$

Notice that once we set $S = N^*$ and $B = N^*\left(\frac{\alpha}{1-\alpha}\cdot\frac{c^s}{c^b}\right)$, then these equations are the FOC of the above auxiliary problem (see proof of Theorem 1). In addition, the Efficient Matching conditions and optimal investment conditions are both implied by the FOC. Therefore, an equilibrium exists.

For assortative matching, the proof of Theorem 2 applies in this case as well. The reason is that the proof uses the Constant Surplus equations of only one side of the market at a time, and thus the same argument holds even when buyers and sellers have different search costs, measures, and bargaining power. □

8.4 Omitted Proofs for Section 3

Claim 1. The NAM allocation is an equilibrium if and only if $c \leq 1/8$. The All Skills Match allocation is an equilibrium if and only if $c \geq 1/8$. The Social Norm allocation is an equilibrium if and only if $l \leq 1$.

Proof.

All Skills Match: By Lemma 2, in any all skills match equilibrium, it must be that $\Delta v = 2 - y$ and $\Delta w = 2 - x$. The Inflow=Outflow equations require that $N = 1$, $x = F(\Delta v) = F(2-y)$ and $y = F(\Delta w) = F(2-x)$. Given that F is uniform with mean $\mu = 1.5$, the only interior solutions are $x = y = 1/2$ and $\Delta v = \Delta w = 3/2$. Values which satisfy the Constant Surplus equations are: $v_1 = w_1 = 17/8 - c$ and $v_0 = w_0 = 5/8 - c$. Such values constitute an equilibrium if and only if the Efficient Matching conditions are satisfied. That is, there is an All Skills Match equilibrium if and only if $v_1 + w_1 = 17/4 - 2c \leq 4$ and $v_0 + w_0 = 5/4 - 2c \leq 1$ which holds if and only if $c \leq 1/8$ (the constraints $v_1 + w_0 \leq 3$ always hold).

Every set of values which satisfies the Constant Surplus equations has that $v_1 + w_1 = 17/4 - 2c$. Therefore, if $c > 1/8$, then the Efficient Matching condition is violated and there is no All Skills Match equilibrium.

NAM: If $c \leq 1/8$, the values $v_1 = w_1 = \frac{3+\mu}{2} - 2c$ and $v_0 = w_0 = \frac{3-\mu}{2} - 2c$ support the NAM allocation as an equilibrium. That is, the tuple $\langle z, M^{NAM}, v, w\rangle$ where the state $x = y = 1/2$ and $N = 2$, and matching $M^{NAM} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is an equilibrium: Inflow=Outflow: The marginal values are $\Delta v = \Delta w = \mu$, and so every type below the median invests. The inflow of skill (unskilled) buyers is 1/2 which equals the outflow, and the same for the sellers.

The Constant Surplus equations hold: $\frac{1}{2}(3 - v_1 - w_0) = \frac{1}{2}(3 - w_1 - v_0) = 2c$,

Efficient Matching conditions hold: since $\mu = 3/2$ as long as $c \leq \frac{1}{8}$,

$$
\begin{aligned}
v_0 + w_0 &= 3 - \mu - 4c > 1 = g_{00} \\
v_1 + w_1 &= 3 + \mu - 4c > 4 = g_{11} \\
v_1 + w_0 &= v_0 + w_1 = 3 - 4c < 3 = g_{10} = g_{01}.
\end{aligned}
$$

If $c > 1/8$, then any set of values that satisfies the Constant Surplus equations with $m_{11} = m_{00} < 1$, must have $3 - v_1 - w_0 = 3 - w_1 - v_0 = 4c \implies v_1 + w_1 + w_0 + v_0 = g_{10} + g_{01} - 8c = 6 - 8c$. However, the RHS$< 5$ when $c > 1/8$, and therefore either $v_1 + w_1 < 4 = g_{11}$ (and skilled agents match) or $v_0 + w_0 < 1 = g_{00}$ (and unskilled agents match).

Social Norm: By Lemma 2, in equilibrium, the marginal values $1 < \Delta v < 2$ and $1 < \Delta w < 2$. Therefore, if the support is large $l > 1$, the Social Norm allocation is not an equilibrium because a positive measure of agents on each side become skilled and unskilled. Conversely, if the support is small $l < 1$, the NAM equilibrium exists, a supporting set of values is $v_1 = 2.5 - c$, $v_0 = 0.5 - c$, $w_1 = 1.5 - c$ and $w_0 = 0.5 - c$.

The buyer with the highest cost wants to invest because her cost $\beta \leq 2 = \Delta v$ and the seller with the lowest cost does not want to invest because her cost $\sigma > \Delta w = 1$. The market clears every period, and so inflow equals outflow.

The Constant Surplus equations are satisfied: $g_{i0} - v_i - w_0 = 2c, \forall i$ and $g_{1j} - v_1 - w_j = 2c, \forall j$.

The Efficient Matching conditions hold: $v_i + w_j \leq g_{ij} \, \forall i, j$, because all pairs (including unchosen skills) generate positive surplus. □