Market Selection and the Information Content of Prices*

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Abstract

We study a large, common-value auction where k_s identical objects of unknown value are auctioned off to n bidders. Buyers choose, based on their private information, between bidding in the auction and an outside option. We first focus on an exogenous outside option that delivers a state-contingent payoff. If the object-to-bidder ratio in the auction exceeds a certain threshold, then there is no equilibrium where the auction price perfectly reveals the true value of the object. Conversely, if the object-to-bidder ratio is less than this threshold, then information is aggregated as the market grows arbitrarily large. We then turn to a model where bidders choose to participate in one of two concurrent auction markets. The outside option for one auction is the equilibrium value of participating in the alternative auction, i.e., outside options are endogenously determined. If frictions lead to uncertain gains from trade in the first auction and the object-to-bidder ratio exceeds a certain cutoff, then information is not aggregated in either market even if the second auction is frictionless. This is because the two auction markets serve as state-contingent outside options for each other.

Keywords: Auctions, Large markets, Information Aggregation.

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1. Introduction

Consider an auction in which k_s identical common-value objects of unknown value are sold to n bidders, each with a unit demand. The sale is conducted through a sealed-bid auction where each of the highest k_s bidders receives an object and pays a uniform price equal to the highest losing bid. In such an auction, if each bidder has an independent signal about the unknown value of the object, then the equilibrium auction price converges to the object's true value as the number of objects and the number of bidders grow arbitrarily large (see Pesendorfer and Swinkels (1997)). Therefore, the auction price reveals the unknown value of the object and thus aggregates all relevant information dispersedly held by the bidders.

Most previous work on information aggregation in auctions takes the distribution of types that bid in the auction as exogenously given.[1] Yet, in many instances, bidders strategically decide whether to trade in a particular market after weighing their alternatives. In other words, the bidder distribution is endogenously determined jointly by the set of available alternatives and the bidders' expectations about the relative attractiveness of these alternatives. Our focus in this paper is a model where bidders choose, based on their private information, between the auction (market s) and an outside option (market r). This framework allows us to highlight the interplay between self-selection into an auction, bidding behavior in the auction, and the information content of prices.

We assume that there are two states that are equally likely a priori and that each object's common value (V), is equal to one in the good state and zero in the bad state. We first focus on an **exogenous** outside option that pays $u(r|V = v)$ if the value of the object is equal to v. Most past work implicitly assumes that a bidder's outside option is equal to zero in each state. In contrast, we assume that the outside option's value is a nondecreasing function of v (monotonicity) and is positive in at least one state (nontriviality). Under these assumptions, we show that if the object-to-bidder ratio ($\kappa_s := k_s/n$) exceeds a certain threshold $\bar{\kappa}_{ex}$, then there is no equilibrium where the auction price perfectly reveals the state as the market grows arbitrarily large. In other words, there is no equilibrium sequence which aggregates information. An implication of this finding is perhaps more instructive. If the expected value of a nontrivial, monotone outside option is equal to zero, then there is no equilibrium where information is aggregated. In this case, a bidder who opts for the outside option faces a loss in the bad state. However, pessimistic bidders can insure against a loss in the bad state by bidding zero in the auction. In fact, more optimistic types choose the outside option, more pessimistic types bid in the auction, and such a pattern of self-selection precludes information aggregation. We provide more intuition for these findings using an illustrative example further below.

Our equilibrium characterization further elucidates the mechanism that leads to prices that fail to aggregate all available information. If the outside option is valuable in both states (i.e., $u(r|V = v) \geq 0$ for all v), then there is a unique equilibrium. In this equilibrium, optimistic types select market s, but the expected number of bidders is less than the number of objects for sale, with positive probability, in both states of the world. In this case, information aggregation fails because of a lack of competition, even though the pattern of self-selection drives the auction price towards the object's value. Alternatively, if the outside option is valuable only in the good state

[1] Papers by Lauermann and Wolinsky (2017) and Murto and Valimaki (2014) are notable exceptions.

(i.e., $u(r|V=0)<0$ and $u(r|V=1)>0$), then there may be many equilibria. In all of these equilibria, the expected number of bidders exceeds the number of objects (i.e., there is sufficient competition), but only the pessimistic types select market s. In this case, information aggregation fails because of the pattern of self-selection, even though there is sufficient competition.

Next, we turn to a model where bidders choose between two concurrent auction markets: the **endogenous** outside option for one is the equilibrium value of participating in the other. In addition to the frictionless auction market s, we assume that there is another, possibly frictional auction market r where there are an additional $n\kappa_r = k_r$ units of the same object on sale and $\kappa_s + \kappa_r < 1$. Our goal here is to better understand markets that generate outside-option payoff profiles that disrupt information aggregation. If there are frictions in market r and the object-to-bidder ratio in market s exceeds a certain cutoff $\bar{\kappa}_{en}$, then there is again no equilibrium where information is aggregated in either market. Frictions in market r inhibit information aggregation in the frictionless market as well, because frictions transform market r into a nontrivial, monotone outside option for market s. In turn, the distribution of types that select market s forces the price to diverge from value in market s, and thus market s also serves as a nontrivial, monotone outside option for market r. In contrast, if there are no frictions in market r, then information is aggregated in both markets in every equilibrium. Therefore, our findings suggest that institutional differences are key for generating outside options that can hinder information aggregation.

We model frictions in market r as a reserve price $c \geq 0$. The reserve price has various interpretations: (1) It is a *reserve price* set by a single auctioneer selling the k_r goods. (2) The auction is comprised of k_r nonstrategic sellers, the reservation value (or the cost) for these sellers is equal to c, and each seller requires at least c in order to participate in the auction, i.e., there are *informational frictions* as in Myerson and Satterthwaite (1983).[2] (3) A *government/regulator imposes a minimum price*. Although we focus on a reserve price, other institutional differences can also be detrimental to information aggregation. In particular, if there is a positive cost $c>0$ for submitting a bid in market r (as in Murto and Valimaki (2014)), or alternatively if market r is a "pay-as-you-bid" auction (as in Jackson and Kremer (2007)) or an "all-pay" auction (as in Chi et al. (2019)), then market r can generate an outside option for market s with consequences similar to the reserve price that we discuss in detail in the paper.

For the case where market r is perturbed by a small friction ($c>0$ but small), we characterize all equilibria. In every equilibrium, the expected prices are equalized across the two markets in each state. Therefore, from the perspective of a bidder who wins an object with probability one, the state-contingent payoffs are also equalized across markets. The pattern of self-selection is the main force that equalizes prices and therefore state-contingent payoffs. A large disparity in the state-contingent payoffs in the two markets would imply that optimistic bidders select the market that generates state-contingent payoffs with higher variance (i.e., large losses in state 0 that are compensated by large gains in state 1) and pessimistic bidders select the option that generates state-contingent payoffs with lower variance. In other words, market selection would have a cutoff structure. However, if market selection has a cutoff structure and an auction attracts the type distribution's upper tail (i.e., the more optimistic types), then we show that the price is driven

[2] The auction is in fact a double auction with nonstrategic sellers who simply bid their valuations c.

towards the object's value, decreasing the variance of state-contingent payoffs. In contrast, if an auction attracts the type distribution's lower tail (i.e., the more pessimistic types), then the price diverges from the object's value, which increases the variance of state-contingent payoffs. Thus, an equilibrium is sustained only if types self-select across markets in a way that equalizes state-contingent payoffs (and prices) in the two markets.

Previous work on information aggregation mainly focuses on homogeneous (or highly correlated) objects that trade in a single centralized, frictionless auction market. However, such a centralized market is an exception rather than the rule. Fragmentation, the disperse trading of the same security in multiple markets, is commonplace: Many stocks listed on the New York Stock Exchange trade concurrently on regional exchanges (see Hasbrouck (1995)). Investors, who participate in a primary treasury bond auction, could purchase a bond with similar cash-flow characteristics from the secondary market. Labor markets are linked but also segmented according to industry, geography, and skill. Buyers in the market for aluminum or steel can choose between the London Metal Exchange or the New York Mercantile Exchange. Such fragmented markets and exchanges also differ in structure, rules and regulations. In particular, markets are heterogeneous in terms of the frictions that participants face. The results that we present in this paper suggest that selection into markets can have important implications for the information content of prices, especially when individuals choose between markets that differ in terms of institutional detail and therefore frictions. In particular, we demonstrate how frictions can disrupt information aggregation not only in the market with frictions but also in frictionless, substitute markets because of how imperfectly informed bidders select across markets.

1.1. An Illustrative Example. Suppose, before bidding in the auction, each bidder receives a signal that perfectly reveals the value of the object with probability $1 - g \geq 0$ and receives an uninformative signal with probability g. A bidder who receives the uninformative signal continues to believe that $V = 1$ with probability $1/2$ while a bidder who receives the perfectly revealing signal knows the object's value. Pesendorfer and Swinkels (1997) showed that the auction price P converges to 1 and 0 when $V = 1$ and $V = 0$, respectively, in such an auction as the number of bidders n and the number of object $n\kappa_s$ grow arbitrarily large for any $g < 1$ under the assumption that all bidders participate in the auction. In other words, Pesendorfer and Swinkels (1997) established that information is aggregated because the auction price converges to the object's value in each state. The assumption that all types participate is innocuous if a non-participating bidder's payoff is at most equal to zero in both states. We will argue that an outside option, which has state contingent payoffs, can hinder information aggregation.

First suppose bidders have access to an **endogenous outside option**, that is, each bidder can either bid in the original auction market s where there are $n\kappa_s$ objects or bid in another auction market r with a reserve price $c \geq 1/2$ where an additional $n\kappa_r$ units of the same object are on sale and $\kappa_r + \kappa_s < 1$. If all bidders receive perfectly informative signals (if $g = 0$), then there is a unique equilibrium in this auction for each n: In state $V = 0$ all bidders bid zero in auction s because there is a positive reserve price in market r. Therefore, the price is equal to zero and c in markets s and r, respectively. In state $V = 1$, the bidders randomize between the two auctions and bid one in the auction that they choose.[3] Since the bidders randomize, they

[3] Bidders bid their value in the auctions since the auctions are $k_s + 1$ (or $k_r + 1$) price auctions.

are indifferent between the two markets in equilibrium. Moreover, the auction price converges to 1 in state $V = 1$ in both markets: the facts that all bidders bid one and $\kappa_r + \kappa_s < 1$ together imply that the price in one of the two markets must converge to one. Since the bidders are indifferent between the two markets, the price must converge to one in both markets. Therefore, the auction price converges to value and perfectly reveals the state exactly as in Pesendorfer and Swinkels (1997).

We will now argue that price cannot converge to value in market s if there are sufficiently many uninformed bidders. Suppose $g \in (1 - \kappa_r, 1)$, $c > 1/2$, and on the way to a contradiction, assume that price converges to value in auction s. No uninformed bidder or any bidder who knows that the state is $V = 0$ would bid in market r in equilibrium because the price in this market is at least $c > 1/2$ in both states. Consider a bidder who knows that the state is $V = 1$. This bidder's payoff from participating in auction s converges to zero because the auction price converges to one in state $V = 1$ by our initial assumption. The price in market r converges to c in both states because $1-g < \kappa_r$ and because only the informed bid in market r.[4] Therefore, any informed bidder will opt for the outside option in state $V = 1$ for sufficiently large n. However, if no bidder other than the uninformed submit nontrivial bids that exceed zero in market s, then all the uninformed bidders would bid $1/2$, i.e., their valuation for the object. Thus, the price cannot converge to one in state $V = 1$, contradicting our initial assumption. In section 5, we characterize the unique equilibrium for this example and show that the auction price perfectly reveals the state if and only if $\kappa_r < 1 - g$.

Suppose instead a bidder chooses between an **exogenous outside option,** which yields $1-c$ if $V = 1$ and $-c$ if $V = 0$; and auction s, where a bidder's payoff is equal to $V - P$ if she wins an object and equal to zero otherwise. The equilibrium price does not reveal the state in this case either. This is easiest to see if the bidders are perfectly informed: any bidder who picks auction s will submit a bid equal to one in state $V = 1$ and will submit a bid equal to zero in state $V = 0$. Thus, the auction price is equal to zero if $V = 0$. In state $V = 1$ the bidders randomize between bidding in the auction and the outside option. The probability with which bidders choose the auction ensures that the expected auction price in state $V = 1$ is equal to c leaving the bidders indifferent between the two alternatives. More precisely, the price is a random variable which is equal to one with probability c and equal to zero with the remaining probability in state $V = 1$. The expected price averages out to c because the auction price is equal to zero if the number of bidders is less than or equal to the number of objects in the auction and this occurs with probability $1 - c$ in equilibrium. A price equal to zero occurs with probability $1 - c$ in state $V = 1$ and with probability equal one in state $V = 0$. Therefore, a price equal to zero does not reveal the state. A price equal to one is fully revealing but only occurs with probability c. More generally, in section 5 we characterize the unique equilibrium for this auction and we show that information is not aggregated in the unique equilibrium.

1.2. Relation to the Literature. Whether prices aggregate information is a central question in economic theory that was first studied in the context of rational expectation models (see, for example, Grossman and Stiglitz (1976, 1980) and Grossman (1981)). This paper, however,

[4] In the uniform price auction that we study, if there are fewer bidders than objects, then the price is equal to the reserve price.

is more closely related to work that studies information aggregation in large, common-value auctions. Wilson (1977) studied common-value, uniform-price auctions with one object for sale, and Milgrom (1981) extended this analysis to any arbitrary number of objects. Both of these papers show that as the number of bidders gets arbitrarily large, price converges to the true value of the object, but only if there are bidders with arbitrarily precise signals about the state of the world. Pesendorfer and Swinkels (1997) further generalized the analysis to the case where there are no arbitrarily precise signals. They showed that prices converge to the true value of a common-value object in all symmetric equilibria if and only if both the number of identical objects and the number of bidders who are not allocated an object grow without bound.[5] Our model is closest to the auction model studied by Pesendorfer and Swinkels (1997).[6]

We make three main contributions to the literature on information aggregation in multi-object common-value auctions. (1) We are the first to study bidding behavior in a multi-object common-value auction where bidders have outside options and the distribution of types is endogenously determined. (2) In this context, we highlight a new mechanism, based on self-selection, that can lead to the failure of information aggregation. (3) We show that information is aggregated if people choose between multiple, **frictionless** auction markets. In this case, the argument of Pesendorfer and Swinkels (1997) does not necessarily apply because the equilibrium bidding function need not be strictly increasing. Nevertheless, using the pattern of self-selection across markets, we establish that information is aggregated in all auction markets.

Our paper is closely related to recent work on single-unit common-value auctions by Lauermann and Wolinsky (2017) and Murto and Valimaki (2014).[7] The novel feature of Lauermann and Wolinsky (2017)'s model is that the auctioneer knows the value of the object but must solicit bidders for the auction, and soliciting bidders is costly. Therefore, the number of bidders in the auction is endogenously determined by the auctioneer. Our paper differs from Lauermann and Wolinsky (2017) because: (1) We study a multi-unit multi-market auction, while they study a single-object single-market auction, and Pesendorfer and Swinkels (1997)'s analysis implies that the information aggregation properties of a multi-unit auctions differ substantially from the information aggregation properties of an auction with a single object. (2) In our model the distribution of types is determined by the participation decision of the bidders, while in their paper the auctioneer's solicitation strategy determines the number of bidders. This implies that in our model participation decisions are type dependent, while in theirs they are type independent but state dependent. In Murto and Valimaki (2014), potential bidders must pay a cost to participate in the auction. This creates type-dependent participation, as in our model. However,

[5] Pesendorfer and Swinkels (2000) generalizes the analysis in Pesendorfer and Swinkels (1997) to a mixed private, common-value environment. Kremer (2002) shows that the information aggregation properties of auctions are more general than the particular mechanisms studied before. See Kremer and Skrzypacz (2005) for related results concerning the link between information aggregation and the properties of order statistics and see Hong and Shum (2004) for a calculation of the rate at which price converges to the true value in large common-value auctions.

[6] There is extensive work on information aggregation and the role of prices in various other market contexts. For example, see Reny and Perry (2006) and Cripps and Swinkels (2006) for large double auctions; Vives (2011) and Rostek and Weretka (2012) for markets for divisible objects; and Wolinsky (1990), Golosov et al. (2014), Ostrovsky (2012), Lauermann and Wolinsky (2015), and Lambert et al. (2018) for search markets and markets with dynamic trading. See also Ellison et al. (2004) and Peivandi and Vohra (2016) for related work.

[7] Also, see Lauermann and Virág (2012) who study the incentives of an auctioneer to release information in a single-unit common-value auction where bidders who do not win an object from the auction receive an outside option.

in contrast to this paper, they focus on a single-object, single-market auction, mainly focus on characterizing equilibria with two bidders and do not emphasize information aggregation.

Lauermann and Wolinsky (2017) and Atakan and Ekmekci (2014) also present models where information aggregation can fail in a large common-value auction. In both of these papers, information aggregation fails because there is an atom in the bid distribution (i.e., many types submit the same pooling bid) and the auction price is equal to this atom (pooling bid) with positive probability in both states of the world. In this paper, although the bid distribution may feature atoms, the failure of information aggregation is not caused by these atoms if the value of the outside option is small. In fact, we show that information aggregation fails either because the auction is not sufficiently competitive[8] or because, although the auction is competitive, the same set of types determine the price in every state due to the pattern of self-selection.[9] In the equilibrium that we construct, we show that the limit-price distribution is atomless over a nontrivial interval of prices that occur with positive probability in both states.

2. Preliminaries

We study an auction where n bidders choose between three mutually exclusive alternatives prior to the auction: 1) A bidder can bid in the auction market s; 2) She can select the outside option r; or 3) She can choose neither and receive a payoff equal to zero. A bidder decides between these alternatives based on her private signal, which provides information about both the value of the object on auction and the value of the outside option. We assume that bidders do not observe anything beyond their private signals when deciding between the alternatives.

Auction s is a common-value, sealed-bid, uniform-price auction with $\lceil \kappa_s n \rceil = k_s$ identical objects on sale, where $\kappa_s \in (0,1)$ is the object-to-bidder ratio.[10] Each bidder has unit demand for the objects and puts value V on a single object, and value 0 on any further objects. Each of the k_s highest bidders receives an object and pays the uniform price P. Thus, a bidder who wins a good at price P enjoys utility $V - P$. A bidder who chooses the auction but fails to win an object receives a payoff equal to zero.

The auction price is equal to the highest losing bid, that is, the $k_s + 1$st highest bid. Ties are broken uniformly and randomly. If many bidders opt for the outside option, then there can be fewer bidders than there are objects in the auction. In this case, i.e., if the number of bidders is less than or equal to the number of goods, the auction price is equal to zero. A bidder does not observe anything beyond her private signal when deciding which bid to choose in the auction.

The value V (or the state of the world) is common across players, but unknown. The value is drawn from $\mathcal{V} = \{0, 1\}$ according to a common prior $\pi = 1/2$.[11]

We consider both an **exogenous** outside option and an **endogenous** outside option. In the exogenous outside-option case, if a bidder chooses the outside option r, then her payoff is equal to $u(r|V)$, i.e., her payoff is an exogenously specified function of the unknown state V.

[8] Information aggregation fails because self-selection into the auction implies that the number of bidders who are not allocated objects is bounded away from infinity with positive probability in both states, i.e., Pesendorfer and Swinkels (1997)'s double-largeness conditions fails with positive probability.

[9] In this case, the number of bidders who are not allocated objects is infinite with probability one in both states, i.e., Pesendorfer and Swinkels (1997)'s double-largeness conditions is satisfied, but information aggregation nevertheless fails.

[10] The smallest integer not less than x is denoted by $\lceil x \rceil$.

[11] We focus on a uniform prior for expositional simplicity only and none of our results depend on this assumption.

The endogenous outside option's value is determined in an auction market that runs parallel to the auction under consideration. We describe the outside options in more detail in the relevant sections.

Prior to deciding between the three available alternatives, each bidder receives a signal $\theta \in [0,1]$ according to a continuous, increasing cumulative distribution function $F(\theta|v)$ that admits a density function $f(\theta|v)$, $v = 0,1$.[12] Conditional on $V = v$, the signals are identically and independently distributed. Given that there are two states of the world, the signals satisfy the monotone likelihood ratio property **(MLRP),** possibly after a reordering of the signal. In other words, the likelihood ratio

$$l(\theta) := \frac{f(\theta|1)}{f(\theta|0)},$$

is a nondecreasing function of θ. Throughout the paper, we maintain the following additional assumptions on the signal distribution:

Assumption 1. ***No uninformative signals****, that is,* $F(\{\theta : l(\theta) = 1\}) = 0$.

Assumption 2. ***Signals contain bounded information****, i.e., there is a constant* $\eta > 0$ *such that* $\eta < l(\theta) < \frac{1}{\eta}$ *for all* $\theta \in [0,1]$.

The first assumption states that mass of signals that contain no information is equal to zero. This is a strengthening of MLRP, but it is weaker than assuming strict MLRP. The second assumption is a technical condition that is also maintained by Pesendorfer and Swinkels (1997). These assumptions significantly simplify the statements and proofs of our results. However, neither of these two assumptions are needed to show that information aggregation fails under the assumptions outlined in the paper. In fact, in our motivating example, neither of these two assumptions are satisfied, but information aggregation nevertheless fails.

2.1. Strategies and Equilibrium. We represent bidder behavior by distributional strategies. A distributional strategy is a measure H on $[0,1] \times \{s, r, \mathit{neither}\} \times [0,\infty)$ with marginal distribution $F(\theta) = (F(\theta|1) + F(\theta|0))/2$ on its first coordinate (see Milgrom and Weber (1985)). A symmetric strategy profile is one in which all players use the same distributional strategy H, and we refer to a symmetric strategy profile simply by strategy H. For a given strategy H, we define the **measure of types in auction** s as $F_s^H(\theta) := H((0,\theta) \times \{s\} \times [0,\infty))$, we define the **selection function** $a^H : [0,1] \to [0,1]$ as the function such that $F_s^H(\theta) = \int_0^\theta a^H(\theta) dF(\theta)$.[13] In words, $a^H(\theta)$ is the probability that type θ bids in auction s. Also, we define the measure of types in market s, conditional on $V = v$, as $F_s^H(\theta|v) := \int_0^\theta a^H(\theta) dF(\theta|v)$, and we set $\bar{F}_s^H(\theta|v) = F_s^H(1|v) - F_s^H(\theta|v)$. We term a symmetric bidding strategy pure if there is a function $b : [0,1] \to [0,\infty)$ such that $H(\{\theta, s, b(\theta)\}_{\theta\in[0,1]}) = H([0,1] \times \{s\} \times [0,\infty)) = F_s^H(1)$. We focus on the symmetric Nash equilibria of the game Γ. In what follows we ignore, without loss

[12]For any half-open interval $(\theta', \theta'']$, we use $F((\theta', \theta'']|v) := F(\theta''|v) - F(\theta'|v)$, i.e., slightly abusing notation, we also use F to denote the measure induced by the cumulative distribution function (through Lebegue's theorem) .

[13]The function a^H is the Radon-Nikodym derivate of F_s^H with respect to F and is unique up to almost every $\theta \in [0,1]$.

of generality, the option of choosing "neither" because this option is never chosen by a positive measure of types in any symmetric equilibrium.[14]

The notation $\Pr^H$ represents the joint probability distribution over states of the world, signal and bid distributions, allocations, market choices, and prices, where this distribution is induced by the symmetric strategy H and the joint distribution of signals and states of the world. We denote the payoff to type θ from bidding b in auction s if players are using strategy H by $u^H(s, b|\theta)$, and type θ's payoff under strategy H by $u^H(\theta)$. The kth highest signal out of n signals is denoted by $Y^n(k)$. Also, we let $Y_s^n(k)$ denote the kth highest type that bids in the auction, and we set $Y_s^n(k)$ equal to zero if there are fewer than k bidders in the auction.

The following lemma allows us to work exclusively with pure and nondecreasing bidding strategies. Moreover, as in Pesendorfer and Swinkels (1997), if the bidding function is increasing over an interval, then any type θ in this interval bids her value conditional on $Y_s^{n-1}(k_s) = \theta$, i.e., conditional on being the pivotal bidder in the auction.

Lemma 2.1. *Any equilibrium H can be represented by a nondecreasing bidding function b^H. Moreover, if $b^H(\theta)$ is increasing over an interval (θ', θ''), then*

$$b^H(\theta) \quad = \quad \mathbb{E}\left[V|Y_s^{n-1}(k_s) = \theta, \theta\right] \tag{2.1}$$

for almost every $\theta \in (\theta', \theta'')$.

Proof. The argument for this lemma follows from Pesendorfer and Swinkels (1997, Lemmata 3-7). $\square$

Below we define a certain type $\theta_s^H(v)$ for each state v such that the expected number of bids above this type's bid in state v, i.e, $\bar{F}_s^H\left(\theta_s^H(v)|v\right)$, is exactly equal to the number of goods in market s. We refer to $\theta_s^H(v)$ as the **pivotal type in state** v because the types that determine the auction price are concentrated around $\theta_s^H(v)$ in a large market by the law of large numbers (see Lemma A.2).

Definition 2.1 (Pivotal types)**.** For any symmetric strategy H, the **pivotal type** in state v is $\theta_s^H(v) := \max\left\{\theta : \bar{F}_s^H(\theta|v) = \kappa_s\right\}$, and $\theta_s^H(v) := 0$ if the set is empty.[15]

For any sequence of strategies $\{H^n\}$, we will denote each $\theta_s^{H^n}(v)$ simply by $\theta_s^n(v)$, and we let $\theta_s(v) = \lim_n \theta_s^n(v)$ and $F_s(\theta|v) = \lim F_s^n(\theta|v)$ whenever such limits exist.[16]

2.2. Definition of Information Aggregation. In order to define information aggregation, we study a sequence of distributional strategies $\mathbf{H} = \{H^n\}_{n=1}^{\infty}$ for a sequence of auctions Γ^n

[14]The reasoning is as follows: if a positive mass of types were to choose "neither" in a symmetric equilibrium, then any bidder who submits a bid equal to zero in auction s would win an object with strictly positive probability in state $V = 1$. Thus, all types who choose "neither" and receive a payoff equal to zero would rather bid zero in the auction and receive a strictly positive expected payoff.

[15]The equation $\bar{F}_s^H(\theta|v) = \kappa_s$ can have multiple solutions if F_s^H is flat over a range of θ. However, the function $\bar{F}_s^H(\theta|v)$ is continuous because it is absolutely continuous with respect to $\bar{F}(\theta|v)$. Hence, the set $\left\{\theta : \bar{F}_s^H(\theta|v) = \kappa_s\right\} \subset [0, 1]$ is compact and has a unique maximal element if it is nonempty.

[16]Such limits always exist along a subsequence. This is because the sequence $\{\theta_s^n(v)\}$ is a subset of $[0, 1]$ and because $\{F_s^n(\theta|v)\}$ is a sequence of nondecreasing, continuous, and bounded functions that has a subsequential limit by Helly's theorem.

where the n^{th} auction has n bidders. We assume that the parameters of the auctions, i.e., (v, u, F, κ_s, π), are constant along the sequence and satisfy all the assumptions that we make.

Suppose that the number of bidders n is large. In this case, the law of large numbers implies that observing the signals of all n bidders conveys precise information about the state of the world. A strategy H^n determines a price P^n for the auction Γ^n given any realization of signals. We say that information is aggregated in the auction if this price also conveys precise information about the state of the world. More precisely, (i) if the likelihood ratio $\frac{\Pr(P^n=p|V=1)}{\Pr(P^n=p|V=0)} = \frac{\Pr(V=1|P^n=p)}{\Pr(V=0|P^n=p)}$ is close to zero (i.e., if it is arbitrarily more probable that we observe such a price p when $V = 0$), then an outsider who observes price p learns that the state is $V = 0$. Alternatively, (ii) if the likelihood ratio $\frac{\Pr(V=1|P^n=p)}{\Pr(V=0|P^n=p)}$ is arbitrarily large, then an outsider who observes price p learns that the state is $V = 1$. If the probability that we observe a price that satisfies either (i) or (ii) is arbitrarily close to one, then we say that the equilibrium sequence aggregates information. Our formal definition of information aggregation is given below:

Definition 2.2. (Kremer (2002) and Atakan and Ekmekci (2014)) A sequence of strategies $\mathbf{H}$ aggregates information if the random variables $\frac{\Pr(V=1|P^n)}{\Pr(V=0|P^n)}$ and $\frac{\Pr(V=0|P^n)}{\Pr(V=1|P^n)}$ converge in probability to zero in state 0 and state 1, respectively. That is, for any $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr\left(P^n \in \left\{p : \frac{\Pr(V = v'|P^n = p)}{\Pr(V = v|P^n = p)} \leq \epsilon\right\}\middle| V = v\right) \quad = \quad 1$$

for $v \neq v'$.

We now derive conditions that are necessary and sufficient for information aggregation. Information aggregation fails if the limit price distribution features an atom that occurs with positive probability in both states because conditional on observing this atom, the likelihood ratio is bounded away from zero. The following definition captures such failures.

Definition 2.3 (Pooling by pivotal types)**.** There is **pooling by pivotal types** along a sequence $\mathbf{H}$ if there is a subsequence of pooling bids $\left\{b_p^{n_k}\right\}$ such that $\lim_{k\to\infty} \Pr\left(P^{n_k} = b_p^{n_k}|V = v\right) > 0$ for $v = 0, 1$. Otherwise, there is **no pooling by pivotal types**.

No pooling by pivotal types is a necessary condition for information aggregation because if it does not hold, then the limit price distribution features an atom that occurs with positive probability in both states.

Information aggregation also fails if the supports of the limit price distributions are the same in the two states. The following definition captures such failures using the distance between the pivotal types.

Definition 2.4 (Distinct and arbitrarily close pivotal types)**.** The pivotal types are **distinct** along a sequence $\mathbf{H}$ if $\lim_{n\to\infty} \sqrt{n}\,|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| \to \infty$ and the pivotal types are **arbitrarily close** along a sequence $\mathbf{H}$ if $\liminf_{n\to\infty} \sqrt{n}\,|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| < \infty$.[17]

Distinct pivotal types are also a necessary condition for information aggregation. To see why, recall that the random variable $Y_s^n(k_s + 1)$ denotes the $k_s + 1$ highest type that bids in the

[17]In Lemma A.8 in the Appendix we show that $\sqrt{n}\,|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| < \infty$ if and only if $\sqrt{n}\,|F_s^n(\theta_s^n(1)|0) - F_s^n(\theta_s^n(0)|0)| < \infty$ utilizing our assumption that there are no arbitrarily informative signals.

auction. The auction clears at the bid of this type because bidding is monotone (Lemma 2.1). The distribution of $Y^n_s(k_s+1)$ in state $V = v$ puts most of the mass within finitely many standard deviations of the pivotal type in state $V = v$ and the standard deviation is equal $\frac{1}{\sqrt{n(1-\kappa_s)\kappa_s}}$. If the pivotal types are arbitrarily close, i.e., if the pivotal types are separated by finitely many standard deviations, then the same set of types determine the price and the supports of the limit price distributions are the same in the two states. Therefore, information cannot be aggregated.

In the following lemma, we further show that these two necessary conditions are also sufficient for information aggregation.

Lemma 2.2. *An equilibrium sequence aggregates information if and only if the pivotal types are distinct and there is no pooling by pivotal types.*

We now provide a sketch of the argument for sufficiency. Pick any type θ that is within finitely many standard deviations of the pivotal type in state $V = 1$ and note that the auction can clear only at the bids of such types in state $V = 1$. Distinctness of the pivotal types implies that type θ is infinitely many standard deviations away from the pivotal type in state $V = 0$. Therefore, if type θ does not bid in an atom, then an outside observer, who observes a price equal to θ's bid, is arbitrarily certain that the state is $V = 1$. On the other hand, suppose that θ bids in an atom, i.e., suppose that the price is equal to θ's bid with positive probability in state $V = 1$. In this case, the probability that that the price is equal to θ's bid in state $V = 0$ is equal to zero because there is no pooling by pivotal types. Therefore, once again, an outside observer, who observes a price equal to θ's bid, is arbitrarily certain that the state is $V = 1$.

We say that **price converges to value** along a sequence $\mathbf{H}$ if P^n converges in probability to 1 when $V = 1$ and converges in probability to 0 when $V = 0$. The following lemma provides a condition under which information aggregation implies that price converges to value.

Lemma 2.3. *If an equilibrium sequence aggregates information and* $\lim \mathbb{E}[P^n] > 0$*, then price converges to value.*

For some intuition, suppose that there is a value $b^* > 0$ that separates the support of the limit-price distribution in state 0 from the support of the limit-price distribution in state 1. The first step of the argument shows that no pooling by pivotal types, distinctness of the pivotal types, and $\lim \mathbb{E}\left[P^n\right] > 0$, together imply that there is indeed such a separating $b^* > 0$. We should point out that no pooling by pivotal types and distinct pivotal types are not together sufficient for the existence of such a separating b^*. In particular, if both pivotal types $\theta^n_s(1)$ and $\theta^n_s(0)$ converge to zero (but at different rates so as to ensure that they are distinct) and $\lim \mathbb{E}\left[P^n\right] = 0$, then a separating $b^* > 0$ does not exist.

The second step in the argument shows that the limit-price distribution's support lies below b^* in state 0 and above b^* in state $V = 1$: Suppose, on the way to a contradiction, that the limit-price distribution's support lies above b^* in state $V = 0$ and below b^* in state $V = 1$. Then, any bidder can ensure that she wins an object only in state $V = 1$ with probability one by submitting a bid equal to b^*. Therefore, any bidder that submits a bid greater than b^* can improve her payoff by instead submitting a bid equal to b^*. So, the limit-price distribution's support cannot lie above b^* in state 0.

The third step in the argument argues that all bids less than b^* must converge to zero, and therefore the price in state 0 must converge to zero: Any type who bids less than b^* never wins if $V = 1$. Therefore, the bid of any such type, and in particular the bid of the pivotal type in state $V = 0$ must converge to zero.

The final step in the argument concludes that the price in state $V = 1$ must converge to one: if the expected price in state 1 is strictly less than one, then the pivotal type in state 0 could improve her payoff by bidding one instead of following her equilibrium strategy. Such a deviation improves her payoff because, if she follows her equilibrium strategy, she never wins an object in state $V = 1$ and receives a payoff equal to zero, while under the deviation she wins an object at a price equal to zero in state $V = 0$ and at a price which is strictly less than one in state $V = 1$ with positive probability.

3. Information Aggregation with an Exogenous Outside Option

In this section, we study an auction where each player chooses between bidding in auction s and an exogenous outside option r. As stated previously, if a bidder selects the exogenous outside option, then her payoff is equal to $u\left(r|V = v\right)$. On the other hand, if a bidder selects auction s, then her payoff is equal to $V - P$ if she wins an object and zero if she does not win an object. We assume $u\left(r|V = 1\right) \leq 1$ and the following:

Assumption 3 (Nontriviality)**.** *The value of the outside option is not less than or equal to zero in both states.*

Assumption 4 (Monotonicity)**.** *The value of the outside option is nondecreasing in v.*

If the value of the outside option is less than or equal to zero in both states, then no type would opt for the outside option in any equilibrium, i.e., the auction is equivalent to an auction without an outside option and information is aggregated by Pesendorfer and Swinkels (1997). Assumption 3 rules out this case. Also, given any $u\left(r|V = 0\right)$, if $u\left(r|V = 1\right)$ is sufficiently larger than one, then no type would select the auction in any equilibrium even if the auction price is equal to zero. If no type selects the auction, then the auction price is equal to zero in both states and information aggregation fails trivially. We require $u\left(r|V = 1\right) \leq 1$ in order to focus attention on the more interesting failures of information aggregation. Assumption 4 is natural in our setting because it is satisfied by the endogenous outside option that we discuss in the next section and because this assumption generates the more interesting dynamics. A prominent example of an outside option that satisfies all of our assumptions is an alternative market where bidders can buy an object, identical to those on auction in market s, at a fixed price equal to $c \geq 0$ as in the illustrative example.

This section's main theorem shows that an exogenous outside option that satisfies Assumptions 3 and 4 disrupts information aggregation if the object-to-bidder ratio (κ_s) exceeds a certain cutoff $\bar{\kappa}_{ex}$, which we formally define further below. Conversely, if the object-to-bidder ratio is less than $\bar{\kappa}_{ex}$, then information is aggregated in every equilibrium in market s. We now introduce some notation in order to define the cutoff $\bar{\kappa}_{ex}$:

For a given type $\theta' < 1$, let $\theta^*\left(\theta'\right)$ denote the unique type $\theta < \theta'$ such that $F\left(\left[\theta, \theta'\right]|0\right) = F\left(\left[\theta, \theta'\right]|1\right)$, and let $\theta^*\left(\theta'\right) = \theta'$ if there is no such $\theta < \theta'$. For some intuition, suppose that types $\theta > \theta'$ opt for the outside option, while types $\theta \leq \theta'$ bid in auction s. In this case, $\theta^*\left(\theta'\right)$ is

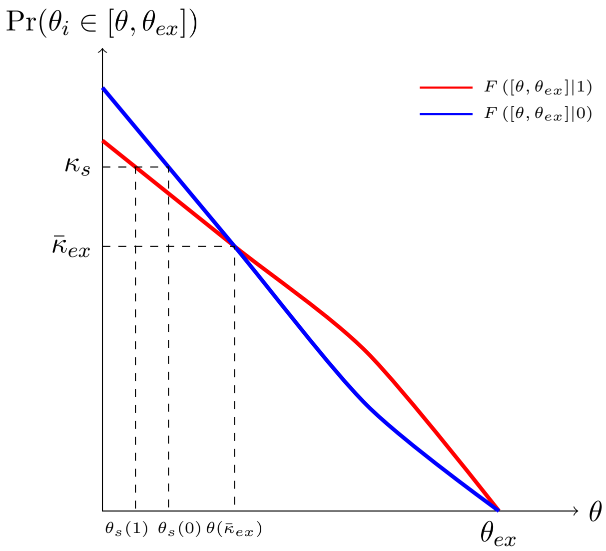


Figure 3.1: The cutoff $\bar{\kappa}_{ex}$. The function $F([\theta, \theta_{ex}]|v)$ depicts the fraction of types above θ that bid in auction s in state v given that all types $\theta > \theta_{ex}$ take the outside option. The cutoff $\bar{\kappa}_{ex}$ is defined as the value of $F([\theta, \theta_{ex}]|v)$ at the point $\theta < \theta_{ex}$ where $F([\theta, \theta_{ex}]|1)$ and $F([\theta, \theta_{ex}]|0)$ cross. If $\kappa_s > \bar{\kappa}_{ex}$, then the pivotal type in state 0 exceeds the pivotal type in state 1. If the functions never cross, then $F([\theta, \theta_{ex}]|0) > F([\theta, \theta_{ex}]|1)$ for all $\theta < \theta_{ex}$ and we then define $\bar{\kappa}_{ex} = 0$. In this case $\theta_s(0) > \theta_s(1)$ for any κ_s.

defined as the type such that the expected number of bidders who bid in auction s with signals that exceed $\theta^*(\theta')$ is the same in both states. The implicit function theorem and MLRP together imply that $\theta^*(\theta')$ is a decreasing function of θ' if $\theta^*(\theta') < \theta'$.[18]

Definition 3.1 (Object-to-bidder cutoff $\bar{\kappa}_{ex}$)**.** Let $\theta_{ex} := \inf\{\theta : \mathbb{E}[u(r|V)|\theta] > 0\}$, and let $\theta_{ex} := 1$ if the set is empty. Define $\bar{\kappa}_{ex} := F([\theta^*(\theta_{ex}), \theta_{ex}]|0) = F([\theta^*(\theta_{ex}), \theta_{ex}]|1)$.

In words, θ_{ex} is the smallest type for which the exogenous outside option is valuable. In fact, the expected value of the outside option is positive for all types that exceed θ_{ex} if the outside option is monotone.[19] The cutoff $\bar{\kappa}_{ex}$ is defined under the assumption that all types that value the outside option (i.e., types $\theta > \theta_{ex}$) choose the outside option and all others bid in the auction. The cutoff $\bar{\kappa}_{ex}$ is chosen so that if the object-to-bidder ratio is equal to $\bar{\kappa}_{ex}$, then the pivotal types in both states are equal to $\theta^*(\theta_{ex})$. Also, see Figure 3.1 for a graphical depiction of $\bar{\kappa}_{ex}$.

The main implication of Definition 3.1 is the following: if the object-to-bidder ratio exceeds $\bar{\kappa}_{ex}$, then the pivotal type in state 0 exceeds the pivotal type in state 1. Such an ordering of pivotal types is ruled out by MLRP if all types participate in the auction. However, if types that exceed $\theta_{ex} < 1$ choose the outside option, then the measure of players participating in the auction is smaller in state 1 than in state 0 as a consequence of MLRP. This implies that $\bar{\kappa}_{ex}$ is less than one. Therefore, there is an open interval $(\bar{\kappa}_{ex}, 1)$ such that whenever the object-to-bidder ratio

[18]For a fixed θ', the functions $F([\theta, \theta']|0)$ and $F([\theta, \theta']|1)$ can only cross once at a type $\theta < \theta'$. This is because $F([\theta, \theta']|0) = F([\theta, \theta']|1)$ if $\theta = \theta'$; $F([\theta, \theta']|0) > F([\theta, \theta']|1)$ if $\theta = 0$ because of MLRP. Moreover, the ratio of the slopes of the two functions, $-l(\theta)$, is non-increasing in θ and $F(\{\theta : l(\theta) = 1\}) = 0$. The implicit function theorem implies that $d\theta^*/d\theta' = f(\theta'|0)(l(\theta') - 1)/f(\theta^*|0)(l(\theta^*) - 1)$. The fact that $F([\theta^*(\theta'), \theta']|0) = F([\theta^*(\theta'), \theta']|1)$ and MLRP together imply that $l(\theta') < 1$. Moreover, if $\theta^*(\theta') < \theta'$, then MLRP implies that $l(\theta^*) > 1$. Therefore, $d\theta^*/d\theta' < 0$.

[19]This is because all types $\theta > \theta_{ex}$ put more weight on state $V = 1$ than θ_{ex} as a consequence of MLRP.

is in this interval, the order of the pivotal types is reversed. The converse is also true, that is, if the object-to-bidder ratio is less than $\bar{\kappa}_{ex}$, then the pivotal type in state 1 exceeds the pivotal type in state 0 even if all types that value the outside option opt for the outside option.

Our main theorem below argues that information cannot be aggregated in any equilibrium if the order of the pivotal types is reversed whenever all bidders who value the outside option opt for the outside option. Conversely, information is aggregated in every equilibrium if the order of the pivotal types is not reversed, even when all the bidders who value the outside option opt for the outside option.

Theorem 3.1. *Assume 3 and 4. If the object-to-bidder ratio is greater than $\bar{\kappa}_{ex}$, then there is no equilibrium sequence that aggregates information. Alternatively, if the object-to-bidder ratio is less than $\bar{\kappa}_{ex}$, then every equilibrium sequence aggregates information.*

Remark 3.1. If $\mathbb{E}\left[u\left(r|V\right)\right] \geq 0$, then $\bar{\kappa}_{ex} = 0$, and therefore there is no equilibrium sequence that aggregates information.

We now provide some intuition for Theorem 3.1. We first argue that information is not aggregated if the object-to-bidder ratio exceeds $\bar{\kappa}_{ex}$. There are two distinct cases to consider: 1) $u\left(r|0\right) < 0 < u\left(r|1\right)$ and 2) $0 \leq u\left(r|0\right) < u\left(r|1\right)$. We will provide intuition for the case where $u\left(r|0\right) < 0 < u\left(r|1\right)$. If $0 \leq u\left(r|0\right) < u\left(r|1\right)$, then the auction has a unique equilibrium for each n. We describe this equilibrium in Proposition 6.1 and provide a discussion after this proposition.

Suppose that $u\left(r|0\right) < 0 < u\left(r|1\right) \leq 1$. In this case, the outside option's value is positive for all types that exceed θ_{ex}. The first step in the argument shows that if information is aggregated, then $\lim \mathbb{E}[P^n] > 0$ and therefore price converges to value (Lemma 2.3). To see why, assume that information is aggregated but $\lim \mathbb{E}[P^n] = 0$. However, then all types would select auction s for all sufficiently large n because any type can guarantee a payoff arbitrarily close to one in state $V = 1$ and zero in state $V = 0$ by submitting a bid equal to one in the auction. However, if all types bid in auction s, then price converges to value and $\lim \mathbb{E}[P^n] = 1/2$ leading to a contradiction.

The second step in the argument notes that if price converges to value in market s, then all types above θ_{ex} would in fact select the outside option because the equilibrium value from bidding in market s is equal to zero. Notice this implies that the more optimistic types select the outside option while the more pessimistic types instead bid in the auction.

The final step in the argument concludes that all types greater than θ_{ex} opting for the outside option, taken together with price converging to value in market s, is incompatible with monotone bidding (Lemma 2.1). More precisely, if all types that exceed θ_{ex} select the outside option, then we find that $\theta_s(0) > \theta_s(1)$ because $\kappa_s > \bar{\kappa}_{ex}$ (see figure 3.1). If information is aggregated in market s, then $\lim_{n\to\infty} b^n\left(\theta_s^n(1)\right) = 1$ and $\lim_{n\to\infty} b^n\left(\theta_s^n(0)\right) = 0$ because price converges to value. However, this leads to a contradiction that proves the result. The findings that $\lim_{n\to\infty} b^n\left(\theta_s^n(1)\right) = 1$, $\lim_{n\to\infty} b^n\left(\theta_s^n(0)\right) = 0$, and $\theta_s(0) > \theta_s(1)$ together contradict that the bidding function is nondecreasing in θ for all n. Intuitively, more pessimistic types opt for the auction and there are more of such types in state $V = 0$. Therefore, the auction clears at the bid of a more pessimistic type in state $V = 1$ than in state $V = 0$ and this is incompatible with price converging to value.

Recall that information is aggregated in an auction if and only if the pivotal types are distinct and they submit distinct bids (no pooling by pivotal types) by Lemma 2.2. Our argument above showed that information aggregation fails whenever $\kappa_s > \bar{\kappa}_{ex}$. Therefore, if $\kappa_s > \bar{\kappa}_{ex}$, then information aggregation must fail either because the pivotal types are arbitrarily close or because the pivotal types bid in an atom. In section 5, we use versions of the illustrative example to construct equilibria where the pivotal types are arbitrarily close and an equilibrium where the pivotal types bid in an atom.

The argument for why information is aggregated if $\kappa_s < \bar{\kappa}_{ex}$ also follows the logic of Lemma 2.2: Note that $\theta_s(1) > \theta_s(0)$ whenever $\kappa_s < \bar{\kappa}_{ex}$, i.e., the pivotal types are distinct. Below, we ague that there is no pooling by pivotal types either whenever $\theta_s(1) > \theta_s(0)$ thus completing the argument.

To sustain a pool, the highest type that submits the pooling bid (denoted by θ_p) must prefer the pooling bid to a slightly higher bid that wins an object with probability one whenever the price is equal to the pooling bid. Also, the lowest type that submits the pooling bid (denoted by $\underline{\theta}_p$) must prefer the pooling bid to a slightly lower bid that avoids winning an object whenever the price is equal to the pooling bid. In other words, pooling must be incentive compatible for type θ_p and individually rational for type $\underline{\theta}_p$. In the terminology of Lauermann and Wolinsky (2017) (or Pesendorfer and Swinkels (1997)), we say that there is **winner's blessing** at pooling if the probability of winning at the pooling bid is higher when $V = 1$ than when $V = 0$, in other words, if a bidder wins more frequently at pooling when the object's value is high. Similarly, there is **loser's blessing** at pooling if a bidder loses more frequently at pooling when the object's value is low. Put another way, if there is loser's and winner's blessing at pooling, then losing is a signal in favor of $V = 0$ and winning a signal in favor of $V = 1$. The strengths of these two forces determine whether a pooling bid is incentive compatible and individually rational. In particular, the loser's blessing's strength determines the lowest pooling bid that is incentive compatible for type θ_p while the winner's blessing's strength determines the highest pooling bid that is individually rational for type $\underline{\theta}_p$. Our key result that establishes that pooling by pivotal types is not possible shows that if $\theta_s(1) > \theta_s(0)$, then there are bounds on the strength of the loser's and winner's blessing at the pooling bid. These bounds preclude a pooling bid that is both individually rational for type $\underline{\theta}_p$ and incentive compatible for type θ_p thus establishing that pooling by pivotal types is incompatible with equilibrium.

We end this section with the following remark which provides some comparative statics for the cutoff object-to-bidder ratio $\bar{\kappa}_{ex}$.

Remark 3.2. Assume 3 and 4. If $u(r|0) \geq 0$, then $\bar{\kappa}_{ex} = 0$. This is because if $u(r|0) \geq 0$, then the outside option is valuable for all types, and so $\theta_{ex} = 0$. On the other hand, if $u(r|0) < 0$, then the cutoff $\bar{\kappa}_{ex}$ is decreasing in $u(r|v)$ for $v = 0, 1$ and increasing in $u(r|1)/u(r|0)$. This is because the type θ_{ex} is decreasing in $u(r|v)$ for $v = 0, 1$ and increasing in $u(r|1)/u(r|0)$. As we stated further above, $\theta^*(\theta_{ex})$ is a decreasing function of θ_{ex}. Consequently, $\theta^*(\theta_{ex})$ is increasing in $u(r|v)$ for $v = 0, 1$ and decreasing in $u(r|1)/u(r|0)$.

4. Information Aggregation with an Endogenous Outside Option

In this section, we study two concurrent auction markets. Our aim is to demonstrate how frictions in one market can endogenously generate state-contingent payoffs that hinder information aggregation in another, frictionless market. In order to do so, we assume that there are $\lceil n\kappa_r \rceil$ objects for sale in market r, in addition to the $\lceil n\kappa_s \rceil$ objects that are on auction in market s, and $\kappa_r + \kappa_s < 1$. The objects for sale in the two markets are identical. The auction format in market r is identical to the auction format in market s, except for a reserve price $c \geq 0$. The following summarizes our assumption on the reserve price.

Assumption 5. *The uniform price in market r is equal to the maximum of c and the highest losing bid in market r if there are more bidders than objects, and the price is equal to c otherwise.*

This section's main theorem shows that market r, which has a reserve price $c > 0$, serves as a nontrivial and monotone outside option for market s. Therefore, if the object-to-bidder ratio in market s exceeds a certain cutoff $\bar{\kappa}_{en}$ (described further below), then the logic of Theorem 3.1 implies that information is not aggregated in market s along any equilibrium sequence. The theorem also considers two other cases: (1) the reserve price in market r is equal to zero and (2) the object-to-bidder ratio in market s is less than $\bar{\kappa}_{en}$. In both of these cases, the theorem argues that information is aggregated in market s along every equilibrium sequence. Before stating our main theorem, below we define the cutoff $\bar{\kappa}_{en}$:

Let $\theta_r^F(1)$ denote the pivotal type in state $V = 1$ if all types were to bid in auction r, that is, $\theta_r^F(1)$ is the unique type that satisfies the equality $1 - F(\theta_r^F(1)|1) = \kappa_r$. Also, recall that $\theta^*(\theta')$ is defined as the type such that the mass of types in the interval $[\theta^*(\theta'), \theta']$ is the same in both states.

Definition 4.1. Let $\theta_{en} := \max\left\{\theta_r^F(1), \inf\{\theta : \Pr(V = 1|\theta) > c\}\right\}$ and $\theta_{en} := 1$ if the set is empty. Define $\bar{\kappa}_{en} := F([\theta^*(\theta_{en}), \theta_{en}]|0) = F([\theta^*(\theta_{en}), \theta_{en}]|1)$.

To better understand the definition, suppose that all types greater than θ_{en} select auction r while all types smaller than θ_{en} bid in auction s. As in Definition 3.1, the cutoff $\bar{\kappa}_{en}$ is defined so that if the object-to-bidder ratio is equal to $\bar{\kappa}_{en}$, then the pivotal types in both states are equal to $\theta^*(\theta_{en})$. Moreover, suppose all types who select market r bid according to a strictly increasing bidding function. Then, type θ_{en} is defined as the smallest type who makes a positive profit in market r in an arbitrarily large auction. To see why the definition captures this property, note that θ_{en} must be at least as large as $\theta_r^F(1)$ because only those types greater than $\theta_r^F(1)$ can actually win an object in the auction in state $V = 1$. Furthermore, any type $\theta > \theta_r^F(1)$ will make a profit in market r only if $\Pr(V = 1|\theta) > c$ because any such type will win an object with probability one in both states and will pay a price which is at least equal to c.

Theorem 4.1. *Assume 5. If $c > 0$ and the object-to-bidder ratio in market s exceeds $\bar{\kappa}_{en}$, then there is no equilibrium sequence that aggregates information in either market. If the object-to-bidder ratio in market s is less than $\bar{\kappa}_{en}$, then information is aggregated in market s along any equilibrium sequence. If $c = 0$, then the information is aggregated in both markets along any equilibrium sequence.*

Remark 4.1. This theorem studies one example of an alternative market that provides an outside option that can hinder information aggregation. There are many other institutional configurations that could result in similar outcomes. Suppose that the alternative market r uses a different auction format and in particular suppose that (1) market r is pay-as-you-bid (discriminatory price) auction as in Jackson and Kremer (2007), where all bidders who win an object from the auction pay their own bid, (2) market r is an all-pay-auction as in Chi et al. (2019), or (3) the auction format is identical to the format considered in this paper, but there is a cost $c > 0$, that each bidder must pay in order to submit a bid in market r as in Murto and Valimaki (2014). The payoff distributions in these alternative specifications for the outside option have similar properties to the payoff distribution in market r as described by Theorem 4.1: payoffs are negative in state $V = 0$ and positive in state $V = 1$. Therefore, our analysis suggests that information would not be aggregated in market s because only certain types would select auction market s.

Remark 4.2. This theorem shows that an outside observer would not learn the state with certainty after observing the prices in the two markets separately. In subsection 6.3, we further argue that an outside observer would not learn the state with certainty even if she observed the price in both markets.

We now provide some intuition for why information aggregation fails in both markets if $c > 0$ and and the object-to-bidder ratio in market s is greater than $\bar{\kappa}_{en}$. We first argue that information cannot be aggregated in market s. As in the case of an exogenous outside option, if information is aggregated in market s, then price converges to value. This follows from Lemma 2.3because $\lim \mathbb{E}\left[P_s^n\right] > 0$ along any equilibrium sequence [20]

The failure of information aggregation in market s is driven by two main forces: (1) Market r provides a nontrivial, monotone outside option for market s. This is because the payoff for any bidder who chooses market r is at most $-c$ if $V = 0$. (2) Our assumption that $\kappa_s > \bar{\kappa}_{en}$ implies that the pivotal type in state $V = 0$ exceeds the pivotal type in state $V = 1$ if all the types that value the outside option provided by market r in fact choose market r.

Intuitively, information aggregation fails in market s, even though this market is frictionless, because bidders with lower signals, i.e., more pessimistic bidders, self select into market s. This, in turn, implies that there are many more bidders who are willing to pay at least the bid of $\theta_s(1)$ in the bad state, i.e., the demand for goods is higher in the bad state. The fact that the demand is high at the bid of $\theta_s(1)$ exactly when people do not value the goods implies that market s cannot clear properly. Thus, information is not aggregated in this market because this selection effect overwhelms competitive forces.

More precisely, recall we argued that information is not aggregated in market s if $\theta_s(1) < \theta_s(0)$. Suppose that $\kappa_s > \bar{\kappa}_{en}$ and, on the way to a contradiction, assume that price converges to value in market s and therefore the payoff of any type that bids in market s is equal to zero. If this is so, then all types that exceed θ_{en} would opt for market r: To see this, observe that if any type $\theta > \theta_{en}$ did not choose market r, then less optimistic types would not choose market r either. Moreover, at the limit, types that exceed θ_{en} face a choice between market s, where their payoff is equal to zero, and market r, where their payoff is positive (in fact, their payoff is equal to $-c$ if $V = 0$ and $1 - c$ if $V = 1$). This choice is similar to the choice that bidders face when the outside option is exogenous, as in Theorem 3.1. However, if all types that exceed θ_{en} opt for market r and if $\kappa_s > \bar{\kappa}_{en}$, then we find $\theta_s(0) > \theta_s(1)$ (see figure 3.1) and information is not be aggregated in market s.

[20] If $\lim \mathbb{E}\left[P_s^n\right] = 0$, then all bidders would prefer market s for sufficiently large n since the price in market r is at least c by assumption. However, if all bidders opt for market s, then price converges to value in market s and therefore $\lim \mathbb{E}\left[P_s^n\right] = 1/2$.

We now turn our attention to market r. In this market information aggregation fails for any κ_s in contrast to market s where information aggregation fails only if $\kappa_s > \bar{\kappa}_{en}$. The expected price in market r is at least c because of the reserve price. In this case, Lemma 2.3 implies that price converges to one in state $V = 1$ if information is aggregated. However, if price converges to one in state $V = 1$, then the payoff from bidding in market r is negative for all types and therefore no type would choose this market. However, if no type choose this market, then the price is equal to c in both states and information is not aggregated in market r either.

To see why information is aggregated in market s if the object-to-bidder ratio in market s is less than than $\bar{\kappa}_{en}$, recall that we argued information is aggregated in market s if $\theta_s(1) > \theta_s(0)$ in the discussion that followed Theorem 3.1. However, if $\kappa_s < \bar{\kappa}_{en}$, then we find $\theta_s(1) > \theta_s(0)$ even if all bidders that exceed θ_{en} in fact opt for market r. Therefore, we conclude that information is aggregated in market s in every equilibrium if $\kappa_s < \bar{\kappa}_{en}$.

A summary for why information is aggregated (and price converges to value) in both markets if $c = 0$ is as follows: We must have $\theta_m(1) > \theta_m(0)$ for at least some $m \in \{s, r\}$. Therefore, information is aggregated in at least one of the two markets. Suppose, without loss of generality, that $\theta_s(1) > \theta_s(0)$, that is, price converges to value in market s. If price converges to value in market s, then any bidder's payoff in market s is equal to zero. The fact that information is not aggregated in market r means that the price is greater than zero with positive probability if $V = 0$ because otherwise no type would choose market s. Thus, the payoff in market r is strictly negative if $V = 0$. But then a single-crossing property implies that all types that exceed $\theta_r(1)$ would opt for market r. Consequently, applying MLRP, we establish that $\bar{F}(\theta_r(1)|1) > \bar{F}(\theta_r(1)|0)$. This, however, contradicts the fact that $\theta_r(1) \leq \theta_r(0)$ because $\bar{F}(\theta_r(1)|1) = \kappa_r$ implies that $\bar{F}(\theta_r(1)|0) < \kappa_r$, and thus $\theta_r(1) > \theta_r(0)$. However, then information is also aggregated in market r.

We end this section with the following remark which provides some the comparative statics for the cutoff object-to-bidder ratio $\bar{\kappa}_{en}$.

Remark 4.3. The cutoff $\bar{\kappa}_{en}$ is increasing in c and non-increasing in κ_r . This is because the type θ_{en} is increasing in c and non-increasing in κ_r, and, consequently, $\theta^*(\theta_{en})$ is decreasing in c and nonincreasing in κ_r. Moreover, if no type finds it profitable to purchase an object at a price equal to c, i.e., if $c > \Pr(V = 1|\theta)$ for all θ, then $\theta^*(\theta_{en}) = 1$ and $\bar{\kappa}_{en} = 1$.

5. Analysis of the Illustrative Example

Theorems 3.1 and 4.1 showed that information is not aggregated in market s whenever κ_s is sufficiently large. However, the proofs of these theorems are by contradiction and so do not convey the equilibrium mechanisms that deliver non-revealing prices. In this section, we aim to further investigate how information aggregation fails by constructing equilibria that generate prices that do not reveal the state.

In the illustrative example, the outside option's payoff is equal to $V - c$ in the exogenous case and the reserve price in market r is equal to c in the endogenous case. The information structure is given by the following signal structure: Bidders receive signals in three equivalence classes, $\mathcal{E}(0) := [0, 1/3)$, $\mathcal{E}(1/2) := [1/3, 2/3]$, and $\mathcal{E}(1) := (2/3, 1]$ according to the density function given below:

$$f(\theta|V=1)=\begin{cases}0 & \text{for } \theta\in\mathcal{E}\,(0)\\ 3g & \text{for } \theta\in\mathcal{E}\,(1/2)\\ 3\,(1-g) & \text{for } \theta\in\mathcal{E}\,(1)\end{cases}\quad f(\theta|V=0)=\begin{cases}3\left(1-g\frac{(1-q)}{q}\right) & \text{for } \theta\in\mathcal{E}\,(0)\\ 3g\frac{(1-q)}{q} & \text{for } \theta\in\mathcal{E}\,(1/2)\\ 0 & \text{for } \theta\in\mathcal{E}\,(1)\end{cases}\tag{5.1}$$

where $q\in[0,1]$. Bidders that receive a signal $\theta\in\mathcal{E}\,(1)$ know for certain that the state is $V=1$, bidders that receive a signal $\theta\in\mathcal{E}\,(0)$ know for certain that the state is $V=0$, and $\Pr(V=1|\theta)=q$ for bidders that receive a signal $\theta\in\mathcal{E}\,(1/2)$. If $q=1/2$, then types $\theta\in\mathcal{E}\,(1/2)$ are uninformed as in the illustrative example presented in the introduction. On the other hand, if $q>1/2$, then types $\theta\in\mathcal{E}\,(1/2)$ are optimistic, and if $q<1/2$, then they are pessimistic.

If $q=1/2$, then there is a unique equilibrium in the illustrative example, irrespective of whether the outside option is endogenous or exogenous. In this equilibrium, all types in $\mathcal{E}\,(0)$ and $\mathcal{E}\,(1/2)$ as well as a positive measure $x^n:=F^n_s(\mathcal{E}(1)|1)$ of types in $\mathcal{E}\,(1)$ opt for market s. In other words, only the most optimistic types opt for the outside option while the others bid in market s. Equilibrium payoffs ensure that types in $\mathcal{E}\,(1)$ are indifferent between the two options. Any type θ that choose auction s, bids their expected value conditioning on the event that they are pivotal, i.e., $\mathbb{E}\left[V|Y^{n-1}_s(k_s)=\theta,\theta\right]$. In particular, the strictly increasing equilibrium bidding function is atomless and is given by the following expression:

$$b^n_s(\theta,x^n):=\frac{\left(\frac{1-3g\left(\frac{2}{3}-\theta\right)-x^n}{1-3g\left(\frac{2}{3}-\theta\right)}\right)^{n-k_s-1}\left(\frac{3g\left(\frac{2}{3}-\theta\right)+x^n}{3g\left(\frac{2}{3}-\theta\right)}\right)^{k_s-1}}{1+\left(\frac{1-3g\left(\frac{2}{3}-\theta\right)-x^n}{1-3g\left(\frac{2}{3}-\theta\right)}\right)^{n-k_s-1}\left(\frac{3g\left(\frac{2}{3}-\theta\right)+x^n}{3g\left(\frac{2}{3}-\theta\right)}\right)^{k_s-1}}\tag{5.2}$$

for each $\theta\in\mathcal{E}(1/2)$, $b^n_s(\theta,x^n)=0$ for each $\theta\in\mathcal{E}(0)$, and $b^n_s(\theta,x^n)=1$ for each $\theta\in\mathcal{E}(1)$. The proposition below summarizes the unique equilibrium.

Proposition 5.1. *Assume f is given by Equation (5.1), $q=1/2$, and $c\in[1/2,1)$. There is a unique equilibrium for each n. In this equilibrium, a positive measure $x^n=F^n_s(\mathcal{E}(1)|1)\in(0,1-g)$ of types $\theta\in\mathcal{E}(1)$ as well as all other types choose auction s and submit a bid equal to $b^n_s(\theta,x^n)$. If the outside option is endogenous, then types $\theta\in\mathcal{E}(1)$, that bid in auction r, submit a bid equal to one.*

For some intuition, suppose that $c>1/2$. It is straightforward to see that no uninformed type or any type $\theta\in\mathcal{E}(0)$ would choose the outside option because any such type's payoff from choosing the outside option is negative. Let us first focus on the case of an exogenous outside option. The proposition asserts that the bidding function is increasing. This follows from three observations: 1) A positive mass of types $\theta\in\mathcal{E}(1)$ must choose market s for each n. This is so because if no $\theta\in\mathcal{E}(1)$ chose market s, then all the uninformed would submit a bid equal to 1/2. This would imply that any $\theta\in\mathcal{E}(1)$ could obtain a payoff of at least 1/2 by submitting a bid equal to 1 in market s. However, then no type $\theta\in\mathcal{E}(1)$ would opt for the outside option because $c>1/2$ which leads to a contradiction. 2) If a positive mass of types $\theta\in\mathcal{E}(1)$ choose market s, then the uninformed submit increasing bids. Bidding is increasing because the uninformed cannot bid in an atom. In order to prove this, we note that if the uninformed bid in an atom, then there is a loser's curse at this atom. Therefore, any uninformed bidder would prefer to break the atom by bidding slightly more or slightly less. 3) If the uninformed submit strictly increasing bids, then the Lemma 2.1 implies that the bidding function is given by Equation 5.2. The proposition further asserts that the equilibrium is unique. In order to show uniqueness, a technical step argues that the payoff of any type $\theta \in \mathcal{E}(1)$ from bidding in market s is strictly decreasing in the mass of $\theta \in \mathcal{E}(1)$ bidding in market s. Therefore, in the unique equilibrium a positive fraction of types $\theta \in \mathcal{E}(1)$ choose market s. The argument for an endogenous outside option also follows the same three steps.[21]

Even though there is a unique equilibrium in the illustrative example, the asymptotic price distribution takes a different form depending on the parameters of the auction. In the remainder of this section, we study the three different asymptotic outcomes that occur along an equilibrium sequence: 1) *Lack of competition*: In this outcome, there are fewer bidders than objects in auction s with positive probability. This implies that the pivotal types are arbitrarily close and information aggregation fails because the price is equal to zero with positive probability in both states. 2) *Arbitrarily close pivotal types with sufficient competition*: In this outcome, there are more bidders than object in both states but the pivotal types are arbitrarily close. Information aggregation fails because the same set of types determine the price in both states and thus the limit price distributions in the two states have the same support. 3) *Pooling by pivotal types*: In this outcome, the pivotal types are distinct. Information aggregation nevertheless fails because the pivotal types submit the same bid, and the auction price is equal to this pooling bid with positive probability in both states. In section 6, we then show that this classification is general and does not depend on the particulars of the illustrative example.

5.1. Exogenous Outside Option Information aggregation fails irrespective of the object-to-bidder ratio κ_s with an exogenous outside option. Although information aggregation fails irrespective of the object-to-bidder ratio, this ratio nevertheless determines the asymptotic price distribution: if the measure of uninformed types exceeds the object-to-bidder ratio (i.e., $g > \kappa_s$), then information aggregation fails because the pivotal types are arbitrarily close. Alternatively, if the object-to-bidder ratio exceeds the measure of uninformed bidders, then information aggregation fails because of a lack of competition. First, we study the case where measure of uninformed types exceeds the object-to-bidder ratio in the following proposition:

Proposition 5.2. *Assume f is given by Equation (5.1), $q = 1/2$, and $c \in [1/2, 1)$. If $g > \kappa_s$, then the pivotal types are arbitrarily close and the equilibrium sequence satisfies the following:*

i. The measure of types $\theta \in \mathcal{E}(1)$ that select market s converges to zero (i.e., $x^n = F_s^n(\mathcal{E}(1)|1) \to 0$).

ii. If $c > 1/2$, then the price P^n converges in distribution to a random variable P. The distribution functions of $\Pr(P \le p|V = 1)$ and $\Pr(P \le p|V = 0)$ are both atomless and strictly increasing on the interval $[0, 1]$. Moreover, $\lim \mathbb{E}[P^n|V = 1] = c$ and $\lim \mathbb{E}[P_s^n|V = 0] = 1-c$.

[21] In this case to complete the argument for uniqueness, we further show that the payoff from bidding in market r is increasing in the mass of $\theta \in \mathcal{E}(1)$ that bid in market s. Once this is established, the fact that a positive mass of types $\theta \in \mathcal{E}(1)$ choose market s in the unique equilibrium again follows since no $\theta \in \mathcal{E}(1)$ choosing market s is incompatible with equilibrium.

iii. If $c = 1/2$, then the price converges in probability to $1/2$ in both states.

For some intuition, suppose that $c > 1/2$. A simple calculation shows that the mass that separates the two pivotal types is equal to the mass of perfectly informed types that bid in market s in state $V = 1$ (i.e., $\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1) = x^n$). If $x^n\sqrt{n} \to \infty$, then the pivotal types are distinct, there are no atoms in the bid distribution (Proposition 5.1), and therefore price converges to value in market s (Lemmata 2.2 and 2.3). However, this would mean that all types $\theta \in \mathcal{E}(1)$ would prefer the outside option for sufficiently large n. Therefore, this cannot be an equilibrium configuration. If, on the other hand, $x^n\sqrt{n} \to 0$, then a direct computation shows that the bids of all the uninformed, and consequently, the price in market s converge to $1/2$ in both states. However, then all $\theta \in \mathcal{E}(1)$ would prefer to bid in market s. Therefore, this cannot be an equilibrium configuration either. This line of reasoning leads us to conclude that $\lim x^n\sqrt{n} \in (0, 1)$.

Along the equilibrium sequence, types $\theta \in \mathcal{E}(1)$ are indifferent between the two markets. This implies that the expected price converges to c in state $V = 1$. Also, the uninformed types' payoff must converge to zero because there are more uninformed types than there are objects. This observation together with the fact that the price in state $V = 1$ converges to c together imply that the expected price in state $V = 0$ converges to $1 - c$.

The proposition also argues that the limit price distributions in the two states are atomless, strictly increasing and share the same support. To see why, pick a type $\theta^n \in \mathcal{E}(1/2)$ which is z standard deviations away from the pivotal type $\theta_s^n(1)$ in state $V = 1$, i.e., $z = \frac{F_s^n(\theta^n|1) - F_s^n(\theta_s^n(1)|1)}{\sigma^n}$ where $\sigma^n = \frac{1}{\sqrt{n(1-\kappa_s)\kappa_s}}$. This type θ^n is $z + x^n/\sigma^n$ standard deviations away from the pivotal type in state $V = 0$ because the mass between the two pivotal types is equal to x^n. The central limit theorem implies that $Y_s^{n-1}(k_s)$ is asymptotically normal and centered around $\theta_s(v)$ in state $V = v$. Therefore, the bid of type θ^n

$$b_s^n(\theta^n, x^n) = \frac{\frac{\Pr\left(Y_s^{n-1}(k_s)=\theta^n|V=1\right)}{\Pr\left(Y_s^{n-1}(k_s)=\theta^n|V=0\right)}}{1 + \frac{\Pr\left(Y_s^{n-1}(k_s)=\theta^n|V=1\right)}{\Pr\left(Y_s^{n-1}(k_s)=\theta^n|V=0\right)}} \to \frac{\frac{\phi(z)}{\phi(z+x)}}{1 + \frac{\phi(z)}{\phi(z+x)}}$$

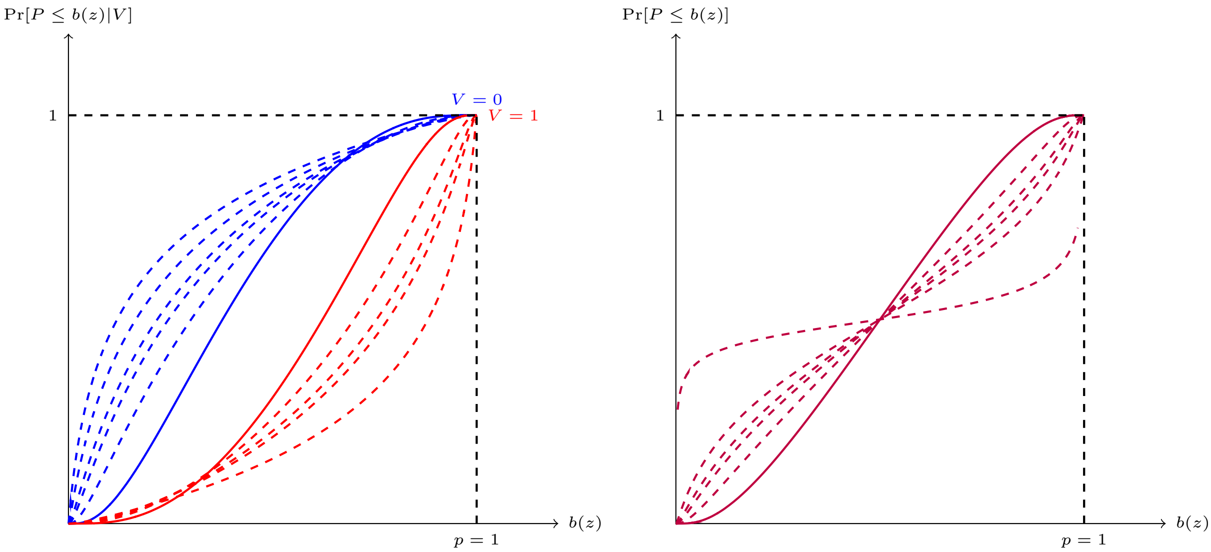
where ϕ is the standard normal density and $x = \lim x^n/\sigma^n$. The limit bid function is increasing in z, converges to 1 as z grows large and converges to zero as z goes to negative infinity. Moreover, the central limit theorem implies that the auction price is less than or equal to the bid of type θ^n with probability $\Phi(z)$ and $\Phi(z + x)$ in states $V = 1$ and $V = 0$, respectively. Also, see Figure 5.1 for a depiction of the limit price distribution.

In the following proposition, we study the case where the expected number of uninformed types is less than the number of goods. In this case, we argue that information aggregation fails because the auction lacks of competition.

Proposition 5.3. *Assume f is given by Equation (5.1), $q = 1/2$, and $c \in [1/2, 1)$. If $g < \kappa_s$, then there is lack of competition in auction s, i.e., $\lim |F_s^n(1|V = 1) - \kappa_s|\sqrt{n} < \infty$ and the equilibrium sequence satisfies the following:*

i. The measure of types $\theta \in \mathcal{E}(1)$ that bid in market s converges to $\kappa_s - g$.

ii. The bids of the uninformed types converge to one, i.e., $b^n(\theta) \to 1$ for all $\theta \in \mathcal{E}(1/2)$.



(a) Limit price distributions in states $V = 0$ (the curves on the left) and $V = 1$ (the curves on the right).

(b) Unconditional limit price distribution.

Figure 5.1: The solid lines give the cumulative price distributions for $c = 0.6$. We implicitly solve for $x = \lim F_s^n(\mathcal{E}(1)|1)/\sigma^n$ using the equilibrium condition $\mathbb{E}[b(z;x)] = c$ where the expectation is taken with respect to the standard normal density. If $c = 0.6$, we numerically find that x is approximately equal to one, i.e., the pivotal types are separated by one standard deviation. The dotted lines give the price distributions as c ranges from 0.6 to 0.8 and therefore as x ranges approximately from 1 to 2. As c approaches one, the outside option's value approaches zero in both states; the price distribution in state $V = 1$ converges to a point mass at $p = 1$; and the price distribution in state $V = 0$ converges to a point mass at $p = 0$. In other words, the price distribution converges to the distribution that perfectly aggregates information.

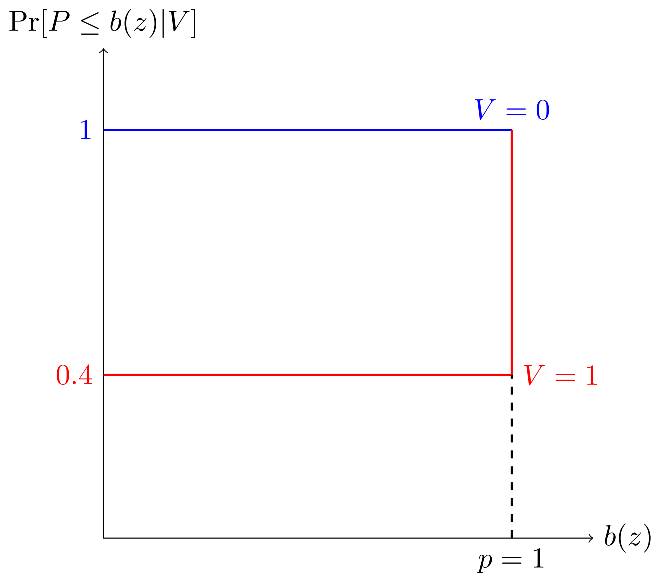


Figure 5.2: Limit price distributions in states $V = 0$ and $V = 1$ for $c = 0.6$. The cumulative distribution in state $V = 0$ jumps from 0 to 1 at $G = 0$. The cumulative distribution in state $V = 1$, jumps from 0 to 0.4 at $P = 0$ and then jumps from 0.4 to 1 at $P = 1$. A price equal to one is perfectly revealing. On the other hand, if an outside observer sees a price equal to zero, then her posterior belief that the state is $V = 1$ is equal to 0.29.

iii. The price in state $V = 0$ converges to zero almost surely. The price in state $V = 1$ converges to a random variable which is equal to zero with probability $\Pr(P = 0|V = 1) = 1 - c$ and equal to one with the remaining probability. The price distribution ensures that all types $\theta \in \mathcal{E}(1)$ are indifferent between bidding in auction s and the outside option.

In words, the above proposition shows that, the measure of types that submit a nontrivial bid that exceeds zero (i.e., types in $\mathcal{E}(1)$ and $\mathcal{E}(1/2)$) is the less the number of object, with positive probability, in both states. The measure of such types is strictly less that κ_s in state $V = 0$ and therefore the price converges to zero in this state. In state $V = 1$, the measure of types in $\mathcal{E}(1)$ and $\mathcal{E}(1/2)$ approaches κ_s at a rate such that the price is equal to zero with probability $1 - c$. Moreover, the bids of all types in $\mathcal{E}(1)$ and $\mathcal{E}(1/2)$ converge to one. This is because the auction never clears at the bids of these types in state $V = 0$. The proof of the proposition is not technical and therefore we provide it in the main text further below. Also, see Figure 5.2 for a depiction of the limit price distribution.

Remark 5.1. If $g = 0$, i.e., if all the bidders are perfectly informed, then the unique equilibrium in the auction is again described by the proposition above and information is not aggregated. It is worthwhile to note that this finding does not depend on the particulars of the illustrative example: if all bidders are perfectly informed, then information aggregation fails with any non-trivial and monotone exogenous outside option. In particular, in the unique equilibrium, if $V = 0$, then the price is equal to zero. If $V = 1$, then the price is a random variable which is equal to zero with probability $1 - c > 0$ in the illustrative example (with probability $u(r|V = 1)$ more generally) and equal to one with the remaining probability. Hence, imperfect information is a necessary condition for information aggregation with an exogenous outside option. In Proposition 5.2 further below, we provide an example of signal structure under which information is aggregated with an exogenous outside option.

Proof. If the measure of types $\theta \in \mathcal{E}(1)$, which bid in market s, is less than $\kappa_s - g$, then the expected number of bidders in state $V = 1$ is less than the number of objects. Therefore, the price in state $V = 1$ converges to zero. However, then all types $\theta \in \mathcal{E}(1)$ would prefer market s. If, on the other hand, the measure of types $\theta \in \mathcal{E}(1)$, which bid in market s, is greater than $\kappa_s - g$, then the pivotal types are distinct and a straightforward calculation shows that price converges to value. But then no type $\theta \in \mathcal{E}(1)$ would bid in market s for sufficiently large n. Therefore, $\lim F_s(\mathcal{E}(1)|V = 1) = \kappa_s - g$.

Next, the proposition asserts that any uninformed type's bid converges to one. A direct computation using the bid function, the fact that $g < \kappa_s$, and $\lim F_s(\mathcal{E}(1)|V = 1) = \kappa_s - g$ together deliver $\lim b^n(\theta) \geq \lim_n \frac{\left(\frac{1-\kappa_s}{1-g}\right)^{n-k_s-1}\left(\frac{\kappa_s}{g}\right)^{k_s-1}}{1+\left(\frac{1-\kappa_s}{1-g}\right)^{n-k_s-1}\left(\frac{\kappa_s}{g}\right)^{k_s-1}} = 1$ for any $\theta > 1/3$.

The proposition also asserts that the limit price has a two point distribution in state $V = 1$. In order to provide types $\theta \in \mathcal{E}(1)$ with an incentive to bid in market s, the expected price must converge to c in state $V = 1$. However, the bids of all types, who are in the market in state $V = 1$, converge to one. An expected price equal to c is sustained in equilibrium by choosing the mass of types $\theta \in \mathcal{E}(1)$ in market s such that the probability of the number of objects exceeding the number of bidders converges to $1-c$ and therefore the price is equal to zero with probability $1-c$ at the limit. However, if $\lim \Pr(P^n = 0|V = 1) \in (0, 1)$, then $\lim |F_s^n(1|V = 1) - \kappa_s|\sqrt{n} < \infty$.[22]

We complete our discussion of the exogenous outside option by calculating the cutoff object-to-bidder ratio $\bar{\kappa}_{ex}$, which is used to state Theorem 3.1, in the remark below: □

Remark 5.2. Recall that θ_{ex} (see Definition 3.1) is the lowest type for whom the outside option's expected payoff is strictly positive. The outside option is strictly profitable for only the types that know that the state is $V = 1$, i.e., types $\theta \in \mathcal{E}(1)$. Therefore, $\theta_{ex} = 2/3$. Continuing along this line, if all types $\theta \geq 2/3$ indeed choose the outside option, then the pivotal types' order is (weakly) reversed, i.e., $\theta_s(1) \leq \theta_s(0)$ for any choice of κ_s. Therefore, $\bar{\kappa}_{ex} = 0$ and for any $\kappa_s > \bar{\kappa}_{ex} = 0$ information aggregation fails along the unique equilibrium sequence.

5.2. Endogenous Outside Option. In contrast to the exogenous outside option case, the object-to-bidder ratios in the two markets determine whether information is aggregated with an endogenous outside option. The next proposition shows that information is aggregated in the unique equilibrium sequence if there are more perfectly informed agents in expectation than there are objects in market r. Alternatively, if there are fewer perfectly informed agents in expectation than objects in market r, then information aggregation fails in both markets.

Proposition 5.4. *Assume f is given by Equation (5.1), $q = 1/2$, and $c \in [1/2, 1)$. If $1-g > \kappa_r$, then information is aggregated in both markets along the unique equilibrium sequence. If $1-g < \kappa_r$, then the unique equilibrium sequence satisfies the following:*

i. The pivotal types in market s are arbitrarily close.

[22] This is because if $\lim (F_s^n(1|V = 1) - \kappa_s)\sqrt{n} = \infty$, then $\lim \Pr(P^n = 0|V = 1) = 0$; and if $\lim (F_s^n(1|V = 1) - \kappa_s)\sqrt{n} = -\infty$, then $\lim \Pr(P^n = 0|V = 1) = 1$.

ii. There is lack of competition in market r, i.e., $F_r(1|1) < \kappa_r$ *and* $F_r(1|0) < \kappa_r$*, and therefore the price in market r converges to c in both states.*

iii. The measure of types $\theta \in \mathcal{E}(1)$ *that select market s converges to zero (i.e.,* $x^n = F_s^n(\mathcal{E}(1)|1) \to 0$*).*

iv. If $c > 1/2$*, then the price in market s converges in distribution to a random variable* P_s*. The distribution functions* $\Pr[P_s \leq p|V = 1]$ *and* $\Pr[P_s \leq p|V = 0]$ *are both atomless and strictly increasing on the interval* $[0, 1]$*. Moreover, the expected prices in state* $V = 1$ *are equalized across the two markets, i.e.,* $\lim \mathbb{E}[P_s^n|V = 1] = c$*.*

v. If $c = 1/2$*, then the price in market s converges to* $1/2$ *in both states.*

The proposition above also describes how information aggregation fails when $1 - g < \kappa_r$: in market r the price is equal to c in both states due to lack of competition. In contrast, there are more bidders than objects in market s. In this market, information aggregation fails because the pivotal types are arbitrarily close. In fact, the limit price distribution in market s is identical to the price distribution that we found in Proposition 5.2. See Figure 5.1 for a graphical depiction of the limit price distribution in market s.

Proof. Suppose the measure of types $\theta \in \mathcal{E}(1)$ is less than κ_r. Since only types $\theta \in \mathcal{E}(1)$ select market r and the measure of such types is less than κ_r, the price in market r converges to c in both states. Hence, the limit payoffs in market r are identical to the payoff of the outside option in the exogenous case. The remainder of the equilibrium characterization therefore follows from the argument for Proposition 5.2.

Alternatively, if the measure of types $\theta \in \mathcal{E}(1)$ exceeds κ_r, then the proposition claims that information is aggregated in both markets. To establish this result, we will argue that the price must converge to one in state $V = 1$ in one of the two markets. This will imply that price converges to one in state $V = 1$ in both markets because along the unique equilibrium sequence a positive mass of types $\theta \in \mathcal{E}(1)$ are active in both markets and these types must be indifferent between the two markets for each n. Suppose that a positive mass of types $\theta \in \mathcal{E}(1)$ select market s, then the pivotal types are distinct in market s. Also, the equilibrium bidding function is strictly increasing, i.e., there is no pooling by pivotal types, by Proposition 5.1. Therefore, price converges to value in market s. Alternatively, suppose that $\lim F_s^n(\mathcal{E}(1)|1) = 0$. In this case, $\lim F_r^n(\mathcal{E}(1)|1) = 1 - g > \kappa_r$, i.e., the expected number of bidders in market r exceeds the number of objects in state $V = 1$. However, then the price in market r converges to one in state $V = 1$ because all the bidders in this market are perfectly informed and submit a bid equal to one. To complete the argument, note that if the price converges to value in state $V = 1$, then it must be the case that $\sqrt{n}F_s^n(\mathcal{E}(1)|1) \to \infty$ and if $\sqrt{n}F_s^n(\mathcal{E}(1)|1) \to \infty$, then the pivotal types are distinct and so price converges to zero in state $V = 0$ in market s also. □

In the following two remarks, we place the illustrative example in the framework introduced to state Theorem 4.1 by calculating the cutoff object-to-bidder ratio that determines whether information is aggregated:

Remark 5.3. Recall that θ_{en} is the lowest type for whom the outside option's expected payoff is strictly positive unless the measure of types above θ_{en} is greater than κ_r in state $V = 1$. Otherwise, θ_{en} is the type with exactly κ_r measure of types above θ_{en} in state $V = 1$ (see Definition 4.1). The outside option is strictly profitable for only the types that know that the state is $V = 1$, i.e., types $\theta \geq 2/3$. Therefore, if $1 - g < \kappa_r$, then $\theta_{en} = 2/3$ because the measure of types above $2/3$ in state $V = 1$ is equal to $1 - g$ which is less than κ_r. On the other hand, if $1 - g > \kappa_r$, then θ_{en} is the type such that $\bar{F}(\theta_{en}|1) = \kappa_r$. Using the density function, we obtain $\theta_{en} = 1 - \frac{\kappa_r}{3(1-g)}$. The cutoff $\bar{\kappa}_{en}$ is calculated under the assumption that $\theta \geq \theta_{en}$ select market r by definition. However, if $1 - g < \kappa_r$ and if all types $\theta \geq \theta_{en}$ select market r, then $\theta_s(1) \leq \theta_s(0)$ for any choice of κ_s because there are no types $\theta \in \mathcal{E}(1)$ that bid in market s. Therefore, $\bar{\kappa}_{en} = 0$. In this case information aggregation fails along the unique equilibrium sequence for any choice of $\kappa_s > 0$. If, on the other hand, $1 - g > \kappa_r$, then $\bar{\kappa}_{en} = 1 - \kappa_r$ and information is aggregated along the unique equilibrium sequence for any $\kappa_s < 1 - \kappa_r$. This is because a measure $1 - g - \kappa_r$ of types in $\mathcal{E}(1)$ bid in market s by definition and this implies that $\theta_s(1) > \theta_s(0)$ if $\kappa_s < 1 - \kappa_r$.

Remark 5.4. If $g = 0$, i.e., if all the bidders are perfectly informed, then the unique equilibrium in the auction is again described by the proposition above and information is aggregated along the unique equilibrium sequence because $g = 0$ implies that $1 - g > \kappa_r$ and therefore $\theta_s(1) \in \mathcal{E}(1) > \theta_s(0) \in \mathcal{E}(0)$ whenever $\kappa_s < 1 - \kappa_r$. However, $\kappa_s + \kappa_r < 1$ by assumption and the fact that all types in $\mathcal{E}(1)$ bid one and all types in $\mathcal{E}(0)$ bid zero together imply that information is aggregated.[23]

5.3. Equilibria with uninformative and perfectly revealing prices. In this subsection, we complete our discussion of the illustrate example by constructing two equilibria: one in which there is pooling by pivotal types with uninformative prices; and another in which information is aggregated with an exogenous outside option. In order to construct an equilibrium where there is pooling by pivotal types, we assume that types $\theta \in \mathcal{E}(1/2)$ are pessimistic (i.e., $q < 1/2$) and in order to construct an equilibrium with information aggregation we assume that types $\theta \in \mathcal{E}(1/2)$ are optimistic (i.e., $q > 1/2$) .

Example 5.1. Assume that f is given by Equation 5.1. Suppose that $0 < q < c < \frac{1}{2}$ and $\kappa_s < g$. Assume that the outside option is exogenous or assume that the outside option is endogenous and $\kappa_r > 1 - g$. Then there exists an $\epsilon > 0$ such that, for all sufficiently large n, there is an equilibrium where all types $\theta \in \mathcal{E}(1)$ select the outside option and all types $\theta \in \mathcal{E}(1/2)$ submit the same pooling bid $b_p = c + \epsilon$ in market s. In this equilibrium, the price in market s is equal to b_p with probability converging to one in both states.

In the equilibrium above, types $\theta \in \mathcal{E}(1/2)$ are willing to submit the pooling bid because the probability of winning at the pooling bid is greater in state $V = 1$ than state $V = 0$. In fact, the posterior of a type $\theta \in \mathcal{E}(1/2)$, conditional on winning an object at the pooling bid, converges to $1/2$ as the market grows large. Consequently, such types make a profit by bidding at the pooling

[23] In contrast to the case with an endogenous outside option, information is not aggregated even when $g = 0$ with an exogenous outside option. This is because the outside option remains valuable for all types in $\mathcal{E}(1)$ irrespective of the measure of types in $\mathcal{E}(1)$ that select the outside option. In contrast, with an endogenous outside option if the measure of types in $\mathcal{E}(1)$ that select the outside option exceeds κ_r, then the price in market r converges to one in state $V = 1$ rendering the outside option valueless.

bid because $b_p < 1/2$. Also, these types would neither want to outbid the pooling bid not take the outside option. If a type $\theta \in \mathcal{E}(1/2)$ outbids the pooling bid, then she always wins and hence her posterior, conditional on winning, is equal to $q < b_p$, i.e., she makes a loss. Similarly, such a type makes a loss if she takes the outside option because $q < c$. No type $\theta \in \mathcal{E}(1)$ bids in market s because the outside option's payoff is equal to $1 - c$ which exceeds $1 - b_p$.

In the following example we assume that types $\theta \in \mathcal{E}\,(1/2)$ are optimistic. In this example, price converges to value along any equilibrium sequence.

Example 5.2. Assume that f is given by Equation 5.1 and the outside option is exogenous. If $\frac{1}{2} < q < c < 1$ and $\kappa_s < g$, then price converges to value along any equilibrium sequence.

As we stated after after Proposition 5.3 information is not aggregated with an exogenous outside option in any equilibrium if all types are perfectly informed. Above we present an example where information is aggregated. In this example, the pivotal types are ordered and distinct for any $F_s^n(\mathcal{E}(1)|1)$.[24] Moreover, all types $\theta \in \mathcal{E}\,(1/2)$ bid according to the increasing bidding function in any equilibrium, i.e., there cannot be pooling. Therefore, price converges to value. This example highlights disperse information as a key ingredient for information aggregation with an exogenous outside option. The proof of uniqueness follows from the same argument as in Proposition 5.1. The fact that information is aggregated then follows from the fact that the pivotal types are distinct for any $\kappa_s < g$.

6. Equilibrium Causes for Non-informative Prices

In this section we show that the three limit price distributions, which we presented in the context of the illustrative example, arise more generally when prices do not aggregate information.

6.1. Lack of Competition.

First, we study an exogenous outside option that is valuable for all bidder types in the following proposition. Under this assumption, we show that information aggregation fails along the unique equilibrium sequence because of a lack of competition.

Proposition 6.1. *Assume 3 and 4. If $u\,(r|0) \geq 0$, then the auction has a unique equilibrium for all n. In the unique equilibrium sequence, the following properties are satisfied:*

i. There is a certain cutoff type $\hat{\theta}^n$ such that all types $\theta < \hat{\theta}^n$ opt for the outside option.

ii. All types $\theta > \hat{\theta}^n$ bid in the auction according to the increasing bidding function $b^n(\theta)$.

iii. All bids converge to one, i.e., $b^n(\theta) \to 1$ for all $\theta > \hat{\theta} = \lim \hat{\theta}^n$.

iv. Moreover, $\lim |F_s^n(1|1) - \kappa_s|\sqrt{n} = x < \infty$ and $F_s(1|0) < \kappa_s$.

v. If $V = 0$, then the price converges to zero almost surely. If $V = 1$, then the price converges to a random variable which is equal to zero with probability $q > 0$ and equal to one with the remaining probability. Moreover, $q = \mathbb{E}\left[u(r|V)|\hat{\theta}\right] \frac{l(\hat{\theta})}{1+l(\hat{\theta})}$ if the function $l\,(\theta)$ is continuous at $\theta = \hat{\theta}$.

It is worthwhile highlighting some properties of the unique equilibrium identified above: All types above a certain cutoff, i.e., the more optimistic types, bid in auction s. Bidders behave as

[24] This is because $q > 1/2$ implies that $\bar{F}_s^n(\theta|V=1) > \bar{F}_s^n(\theta|V=0)$ for any $\theta \geq 1/3$. Moreover, $\theta^n(1) > 1/3$.

they would in the equilibrium of an auction without an outside option, i.e., each type submits a bid which is equal to her value conditional on being pivotal (item *ii*). Therefore, the bid of the pivotal type in state 1 and the bids of all types that exceed $\theta_s(1)$ converge to one. This further implies that if there are many bidders in auction s, then the payoff of these bidders would converge to zero. Hence, only types above the pivotal type $\theta_s(1)$ participate in the auction, but all types who participate bid aggressively (items *iii* and *iv*). In fact, the number of bidders in the auction is less than the number of goods with probability one in state 0 and with probability $q > 0$ in state 1.[25,26] Although information aggregation fails, a price close to one is fully revealing because such a price occurs with positive probability only if $V = 1$.

6.2. Arbitrarily Close Pivotal Types. Next, we study an exogenous outside option that is valuable for only some of the bidder types (i.e., $\bar{\kappa}_{ex} > 0$) in the proposition below. In this proposition, we assume that the outside option's payoff, in absolute value, is less than a certain cutoff

$$\bar{u} := \frac{\kappa_s - \bar{F}\left(\theta_s^F(1)\,|0\right)}{2} > 0, \tag{6.1}$$

where $\theta_s^F(1)$ denotes the pivotal type calculated under the assumption that all types participate in market s.[27] Alternatively, we assume that $u(r|1) < 1/2$ and $u(r|0) \le -1/2$ as in the illustrative example. Under these assumptions, we argue that information aggregation fails because the pivotal types are arbitrarily close even though there is sufficient competition.

Proposition 6.2. *Assume 3 and 4, and suppose that* $\kappa_s > \bar{\kappa}_{ex} > 0$. *If* $|u(r|v)| \le \bar{u}$ *for* $v = 0, 1$ *or if* $0 < u(r|1) < 1/2 \le -u(r|0)$, *then the pivotal types are arbitrarily close and* $F_s(1|v) > \kappa_s$ *for* $v = 0, 1$ *along every equilibrium sequence.*

It is worthwhile highlighting some properties of the equilibria described by the proposition above: In all equilibria, the expected number of bidders who are not allocated an object converges to infinity. Therefore, the failure of information aggregation is not due to insufficient competition.[28] In fact, information aggregation fails because the auction clears at the bids of the same set of types in both states, and this is exactly because the pivotal types are arbitrarily close to each other. Interestingly, and in contrast to Proposition 6.1, this configuration of pivotal types implies that there are no fully revealing prices: any set of prices that occur with positive probability in state 0 also occur with positive probability in state 1 and vice versa. Therefore, the posterior belief of an outside observer remains strictly bounded away from zero and one after ob-

[25]The failure of information aggregation in this equilibrium is related to a failure of the double largeness concept introduced by Pesendorfer and Swinkels (1997). Pesendorfer and Swinkels (1997) showed that information is aggregated if and only if the number of goods and the number of losers not allocated goods converge to infinity. They termed this double largeness. In the unique equilibrium described by Lemma 6.1, double largeness fails with probability one in state $V = 0$, and it fails with positive probability in state $V = 1$.

[26]A similar indeterminacy also arises in the take-over model presented in Ekmekci and Kos (2016), although the decisions are binary in their model.

[27]Note that MLRP implies that $\kappa_s - \bar{F}\left(\theta_s^F(1)\,|0\right) > 0$ and therefore $\bar{u} > 0$. Also note $\kappa_s - \bar{F}\left(\theta_s^F(1)\,|0\right) = \bar{F}\left(\theta_s^F(1)\,|1\right) - \bar{F}\left(\theta_s^F(1)\,|0\right)$, where $\theta_s^F(1)$ is defined implicitly by $\bar{F}\left(\theta_s^F(1)\,|1\right) = \kappa_s$. Therefore, the cutoff $\bar{u}$ is increasing in κ_s if $l\left(\theta_s^F(1)\right) > 1$ and decreasing in κ_s if $l\left(\theta_s^F(1)\right) < 1$. Alternatively, $\bar{u}$ as a function of κ_s is hump-shaped: increasing in κ_s for small values of κ_s and decreasing in κ_s for high values.

[28]If $F_s(1|v) > \kappa_s$ for $v = 0, 1$, then the double largeness condition of Pesendorfer and Swinkels (1997) is satisfied in auction s. Double largeness implies information aggregation if there are no selection effects. Proposition 6.2 shows that information is not aggregated despite the fact that double largeness holds in auction s due to selection effects.

serving the price. Also, see Proposition 5.4 for an example of this equilibrium configuration and Figure 5.1 for a depiction oft the limit price distribution when the pivotal types are arbitrarily close.

A central finding of Proposition 6.2 is that the pivotal types converge to each other as n grows large. The argument is as follows: If the pivotal types do not converge to each other, then two limit outcomes are possible: (1) The pivotal types do not submit the same pooling bid and therefore their bids converge to two distinct values, i.e., $\lim b^n\left(\theta_s^n(1)\right) > \lim b^n\left(\theta_s^n(0)\right)$. However, then information is aggregated, which is incompatible with Theorem 3.1. (2) There is pooling by pivotal types. In this case, a key argument shows that the pooling price must remain larger than a uniform and positive lower bound in order for pooling to be possible. A high pooling price implies that the bidders' loss at the pooling bid exceeds their loss if they choose outside option r in state 0. This, however, implies that market selection has a cutoff structure: optimistic types above a threshold choose market s and all other, more pessimistic, types choose option r. But such a selection of types implies that information is aggregated in market s, which is again incompatible with Theorem 3.1. Ruling out these two alternative outcomes leaves arbitrarily close pivotal types as the only configuration that is compatible with equilibrium.

6.3. Endogenous Outside Options. Finally, we turn our attention to an endogenous outside option in the proposition below. In this proposition, we assume that the reserve price in market r either exceeds 1/2 or is smaller than a particular cutoff $\bar{c}$, which is defined below[29]

$$\frac{\bar{c}}{1-\bar{c}} := \frac{\left(1-\kappa_s-\kappa_r\right)^2}{1-\min\left\{\kappa_s,\kappa_r\right\}+1-\kappa_r-\kappa_s} > 0. \tag{6.2}$$

Under this assumption, we argue that information aggregation fails due to a lack of competition in market r (as in Proposition 6.1) while information aggregation fails in market s because the pivotal types are arbitrarily close even though there is sufficient competition (as in Proposition 6.2).

Proposition 6.3. *Assume 5. For any sequence of equilibria, the measure of types that submit a bid in market r satisfies $F_r(1|0) < F_r(1|1) \leq \kappa_r$, and the price in market r converges to c with probability one if $V=0$ and it converges to a random variable that is equal to c with probability $q>0$ and equal to one with the remaining probability if $V=1$. If $\kappa_s > \bar{\kappa}_{en}$ and $c \notin [\bar{c}, 1/2]$, then the pivotal types in market s are arbitrarily close. If, in addition, $c<\bar{c}$ then the expected prices are equal across states and markets. In particular, $\lim \mathbb{E}\left[P_s^n|V=0\right] = \lim \mathbb{E}\left[P_r^n|V=0\right] = c$, and $\lim \mathbb{E}\left[P_r^n|V=1\right] = \lim \mathbb{E}\left[P_s^n|V=1\right]$.*

Remark 6.1. Theorem 4.1 showed that an outside observer would not learn the state with certainty after observing the prices in the two markets separately. The above characterization further implies that an outside observer could not deduce the state with certainty even if she observed the price in both markets. This is because the price is equal to the reserve price with strictly positive probability in both states in market r, and the support of the price distribution in market s is identical across both states.

[29]Note that this cutoff only depends on the object-to-bidder ratio in the two markets (i.e, it is independent of the signal distribution) and is decreasing in both object-to-bidder ratios.

Proposition 6.1 focused on the case where the reserve price is either less than $\bar{c}$ or greater than 1/2. If the reserve price is not in this range, then the auction that we study can have a variety of different types of equilibria. In particular, Example 5.1 presents an equilibrium under the assumption that the reserve price $c \in [\bar{c}, 1/2]$ where there is pooling by pivotal types in auction s.

7. Discussion and Conclusion

The results that we presented in the paper argued that the price in a large, uniform-price, common-value auction may not aggregate all available information if bidders have access to an outside option whose value is correlated with the common-value object on auction. We showed that exogenous as well as endogenous outside options can hinder information aggregation because of the pattern of type-dependent market selection that such outside options generate. In conclusion, we comment on equilibrium existence in the model that we study and the welfare properties of these equilibria.

7.1. Equilibrium Existence. To the best of our knowledge, equilibrium existence in the framework that we consider is not guaranteed by any result already in the literature. However, if we restrict bids in all markets to a finite grid $B = \{0, \Delta, 2\Delta, ..., \infty\}$, where $\Delta > 0$ is the fineness of the grid so that a symmetric distributional strategy H^{Δ} is a probability measure over $[0,1] \times \{s, r, neither\} \times B$, then equilibrium existence follows immediately from Milgrom and Weber (1985).

7.2. Efficiency. Our focus throughout the paper has been on the informational efficiency of prices, and we have not commented on equilibrium welfare. Typically, welfare considerations are not interesting in common-value auctions because such auctions are efficient as long as all objects are allocated to bidders. This is also true in the auction that we study: given the set of agents that participate in auction s, any allocation of objects to bidders, in which all objects are allocated in state $V = 1$, is efficient. However, the market-selection strategy that players use in equilibrium has welfare implications. In fact, if information is aggregated in market s, then every equilibrium outcome of the model that we study is asymptotically efficient as the market grows arbitrarily large. On the other hand, if information is not aggregated in market s, then the equilibria characterized in Propositions 6.2 and 6.3 remain inefficient even as the market grows large.

In particular, assume an exogenous outside option with $u(r|0) < 0 < u(r|1) < 1$ and consider a planner, with no knowledge of the state or the realized signals, who chooses a market selection strategy $\mu : [0,1] \to \{s, r\}$. That is, the planner chooses s or r for each bidder only as a function of that bidder's type θ. Assume that $F(\theta_{ex}|1) > \kappa_s$. In this case, the planner would optimally choose $\mu(\theta) = r$ for all $\theta > \theta_{ex}$ and $\mu(\theta) = s$ for all $\theta \leq \theta_{ex}$. The findings that we presented as Proposition 6.2 imply that all equilibria are inefficient under the hypothesis of this proposition. In fact, the equilibrium that we construct in Proposition 5.2 is inefficient because there is over-entry into market s. Prices in market s remain bounded away from the object's value because bidders refrain from bidding the price higher in order to avoid winning an object at a high price when $V = 0$. The fact that price remains bounded away from the object's value in market s implies that this market remains profitable for optimistic types. However, this implies that inefficiently many optimistic bidders choose market s instead of market r. In contrast, if

$\kappa_s < \bar{\kappa}_{ex}$, then profits are dissipated in market s because information is aggregated and types $\theta \geq \theta_{ex}$ opt for the outside option in every equilibria. Therefore, such equilibrium outcomes are asymptotically efficient.

The situation is similar for the case of an endogenous outside option. Under the hypotheses of Proposition 6.3, a planner would assign all types $\theta \geq \theta_{en}$ to market r and the remainder to market s. However, Proposition 6.3 shows that this is not how types sort themselves across the two markets in equilibrium, and therefore the equilibrium outcomes are asymptotically inefficient. On the other hand, if $\kappa_s < \bar{\kappa}_{en}$, then Theorem 3.1 argues that information is aggregated in market s and types $\theta \geq \theta_{en}$ opt for market r in every equilibria. Therefore, such equilibrium outcomes are asymptotically efficient.

A. Appendix: Bidding Equilibria

In this section, we define an auction where participation in the auction is exogenously determined by a function $F_s(\cdot)$ that is absolutely continuous with respect to $F(\cdot)$. Given F_s, denote by $\hat{\Gamma}(F_s)$ the auction where each type θ is allowed to bid in the auction with probability $a(\theta)$ and is assigned a payoff equal to zero with the remaining probability $1 - a(\theta)$. The profile H is a bidding equilibrium if it is a Nash equilibrium for the auction $\hat{\Gamma}(F_s)$ where participation is determined by $F_s(\theta)$.

Let $\mathcal{E}(\theta') = \{\theta : l(\theta_i = \theta) = l(\theta_i = \theta')\}$. Each equivalence class $\mathcal{E}(\theta')$ is comprised of types who receive signals that generate the same posterior. If $\mathcal{E}(\theta')$ is not a singleton, then H may involve a range of bids given a signal in $\mathcal{E}(\theta')$. However, for any such H there is another strategy, which is pure and increasing on each $\mathcal{E}(\theta')$, such that this strategy yields the same payoff to the player, and is indistinguishable to any other player. Strategies which differ only in their representation over sets $\mathcal{E}(\theta')$ generate the same joint distribution over values, bids, and equilibrium prices. We choose a representation of H which is pure and nondecreasing over equivalence classes $\mathcal{E}(\theta')$.

Lemma A.1. *Any bidding equilibrium H can be represented by a nondecreasing bidding function b. Moreover, if $b(\theta)$ is increasing over an interval (θ', θ''), then*

$$b(\theta) \quad = \quad \mathbb{E}\left[V|Y_s^{n-1}(k_s) = \theta, \theta\right] \tag{A.1}$$

for almost every $\theta \in (\theta', \theta'')$.

Proof. The argument for this lemma follows from Pesendorfer and Swinkels (1997, Lemmata 3-7). □

The following lemma shows that the bids of the pivotal types determine the auction-clearing price of a sufficiently large auction.

Lemma A.2. *Suppose* $\lim \bar{F}_s^n (0|V = v) > \kappa_s$ *and let* $\underline{\theta}^n$ *denote the type such that* $F_s^n\left([\underline{\theta}^n, \theta_s^n(v)]\,|V = v\right) = \epsilon$ *and* $\underline{\theta}^n = 0$ *if no such type exist. Similarly, let* $\bar{\theta}^n$ *denote the type such that* $F_s^n\left(\left[\theta_s^n(v), \bar{\theta}^n\right]|V = v\right) = \epsilon$ *whenever such a type exists. For every* $\epsilon > 0$,

$$\lim \Pr\left(P^n \in \left[b^n\left(\underline{\theta}^n\right), b^n\left(\bar{\theta}^n\right)\right]|V = v\right) \quad = \quad 1$$

where $b^n(0) = 0$. *Conversely, if* $\lim \bar{F}_s^n (0|V = v) < \kappa_s$, *then* $\lim \Pr\left(P^n = 0|V = v\right) = 1$.

Proof. The law of large numbers implies that $\lim \Pr\left(Y_s^n\left(k_s+1\right) \geq \underline{\theta}^n | V=v\right)=1$ for every $\epsilon>0$. However, if $Y_s^n\left(k_s+1\right) \geq \underline{\theta}^n$, then $P^n=b^n\left(Y_s^n\left(k_s+1\right)\right) \geq b^n\left(\underline{\theta}^n\right)$ because b^n is nondecreasing by Lemma 2.1. Therefore,

$$\Pr\left(P^n \geq b^n\left(\underline{\theta}^n\right) | V=v\right) \geq \Pr\left(Y_s^n\left(k_s+1\right) \geq \underline{\theta}^n | V=v\right),$$

and taking limits proves the first part of the claim. We establish $\lim \Pr\left(Y_s^n\left(k_s+1\right) \leq \bar{\theta}^n | V=v\right)=1$ using the same idea. If $\lim \bar{F}_s^n(0 | V=v)<\kappa_s$, then $\lim \Pr\left(P^n=0 | V=v\right)=1$ also follows directly from the law of large numbers. □

A.1. Pooling Calculations Recall that Lemma 2.2 claims that no pooling by pivotal types and distinct pivotal types are necessary and sufficient for information aggregation. In the next two subsections, we will develop the intermediate results that we will use to show that these two conditions are indeed necessary and sufficient for information aggregation (Lemma 2.2). The results that we present in this subsection allow us to identify when pooling by pivotal types is incompatible with equilibrium.

Given a strategy H, denote by $\Pr\left(b\ wins | P^n=b, V=v, \theta\right)$ the conditional probability that bidder i wins an object with a bid equal to b given that the auction price is equal to b, the state is equal to v and bidder i receives a signal equal to θ. Our assumptions that the signals are conditionally independent given V and that H is symmetric together imply that $\Pr\left(b\ wins | P^n=b, V=v, \theta\right)=\Pr\left(b\ wins | P^n=b, V=v\right)$. This is because once one conditions on the state, the individual signal of bidder i does not provide any additional information (conditional independence). Moreover, this probability is independent of the identity of the bidder that we consider because we focus on symmetric strategies.

Given a pooling bid b_p^n, let $\theta_p^n=\sup\left\{\theta: b^n(\theta)=b_p^n\right\}$, $\underline{\theta}_p^n=\inf\left\{\theta: b^n(\theta)=b_p^n\right\}$, and let $\lim \theta_p^n=\theta_p$ and $\lim \underline{\theta}_p^n=\underline{\theta}_p$ whenever these limits exist. Let the random variables L^n, G^n, and $X^n=L^n+G^n$ denote the number of losers, number of winners (or the number of objects left for the bidders that submit a bid equal to b_p^n), and number of bidders that submit a bid equal to b_p^n, respectively.

Lemma 7 and Corollary 3 in Pesendorfer and Swinkels (1997) shows that pooling cannot be sustained if the bidders are subject to the loser's curse at the pooling bid. More precisely, if

$$\frac{\Pr\left(P^n=b_p^n, b_p^n\ wins | V=1\right)}{\Pr\left(P^n=b_p^n, b_p^n\ wins | V=0\right)}<\frac{\Pr\left(P^n=b_p^n | V=1\right)}{\Pr\left(P^n=b_p^n | V=0\right)}<\frac{\Pr\left(P^n=b_p^n, b_p^n\ loses | V=1\right)}{\Pr\left(P^n=b_p^n, b_p^n\ loses | V=0\right)}, \tag{A.2}$$

then pooling is incompatible with equilibrium.

The following two lemmata allow us to check whether pooling is possible for sufficiently large n. In particular, Lemma A.3 below calculates $\lim \Pr\left(b_p^n\ wins | P^n=b_p^n, V=v\right)$ for a pool that occurs with positive probability in state $V=v$ while Lemma A.4 calculates $\Pr\left(b_p^n\ wins | P^n=b_p^n, V=v\right)$ for a pool that occurs with probability converging to zero in state $V=v$.

The following lemma assumes that the price is equal to the pooling bid with positive probability and shows that $\Pr\left(b_p^n\ wins | P^n=b_p^n, V=v\right)=\mathbb{E}\left[G^n / X^n | P^n=b_p^n, V=v\right]$ converges to $\lim \mathbb{E}\left[G^n | V=v\right] / \mathbb{E}\left[X^n | V=v\right]$ if $\lim F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right] | v\right)>0$. If, on the other hand, $\lim F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right] | v\right)=0$, then the lemma shows that the expected number of objects left for bidders that submit a bid equal to b_p^n (i.e., $\mathbb{E}[G^n|V=v]$) is asymptotically proportional to $n \max\left\{\kappa_s - \bar{F}_s^n\left(\theta_p^n|v\right), 1/\sqrt{n}\right\}$ and $\lim \Pr\left(b_p^n\ wins|P^n = b_p^n, V=v\right)$ is asymptotically proportional to $\lim \mathbb{E}[G^n|V=v]/\mathbb{E}[X^n|V=v]$.

Lemma A.3. *Suppose* $\lim \Pr\left(P^n = b_p^n|V=v\right) > 0$. *Then*

$$\lim \frac{\Pr\left(b_p^n\ wins|P^n = b_p^n, V=v\right)}{\frac{\max\left\{\kappa_s - \bar{F}_s^n\left(\theta_p^n|v\right), \frac{1}{\sqrt{n}}\right\}}{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|v\right)}} \in (0,\infty),$$

$$\lim \frac{\Pr\left(b_p^n\ loses|P^n = b_p^n, V=v\right)}{\frac{\max\left\{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right) - \kappa_s, \frac{1}{\sqrt{n}}\right\}}{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|v\right)}} \in (0,\infty)$$

along any subsequence where the mentioned limits exist. Moreover, if $\lim F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|V=v\right) > 0$*, then*

$$\lim \Pr\left(b_p^n\ wins|P^n = b_p^n, V=v\right) = \lim \frac{\kappa_s - \bar{F}_s^n\left(\theta_p^n|v\right)}{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|v\right)}$$

whenever the limit exists.

Proof. We show if $\lim F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|v\right) > 0$ or if $\lim \Pr\left(P^n = b_p^n|V=v\right) = 1$, then

$$\lim \Pr\left(b_p^n\ wins|P^n = b^n, v\right) = \lim \frac{\kappa_s - \bar{F}_s^n\left(\theta_p^n|v\right)}{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|v\right)}.$$

The proof of the case where $\lim F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|V=v\right) = 0$ is more technical and provided in the online appendix.

Suppose that $Y_s^n(k_s+1) = \theta^n$, $\theta^n \in [\underline{\theta}_p^n, \theta_p^n]$ and $\bar{F}_s^n(\theta^n|v) \in (\kappa_s - \epsilon_1, \kappa_s + \epsilon_1)$. There are k_s bidders with signals above θ^n and the distribution of G^n is binomial, hence $\mathbb{E}[G^n|Y_s^n(k_s+1) = \theta^n, v] := \bar{G}_n = \frac{k_s\left(\bar{F}^n(\theta|v) - \bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\bar{F}^n(\theta|v)}$. Also, $\Pr(G^n < (1-\delta)\bar{G}_n|Y_s^n(k_s+1) = \theta^n, v) \le e^{-\frac{\delta^2}{2}\bar{G}_n}$ for any $\delta \in (0,1)$ by the Chernoff's inequality. Also, $\mathbb{E}[L^n|Y_s^n(k_s+1) = \theta^n, v] := \bar{L}_n = \frac{(n-1-k_s)\left(\bar{F}_s^n\left(\underline{\theta}_p^n|v\right) - \bar{F}_s^n(\theta^n|v)\right)}{1-\bar{F}_s^n(\theta^n|v)} + 1$ because there are $n-1-k_s$ bidders with signals below θ^n and the distribution of L^n is binomial and $\Pr(L^n < (1-\delta)\bar{L}_n|Y_s^n(k_s+1) = \theta, v) \le e^{-\frac{\delta^2}{2}\bar{L}_n}$. The random variable X^n and L^n are independent conditional on $Y_s^n(k_s+1) = \theta^n$. Moreover, $\Pr\left(b_p^n\ wins|Y_s^n(k_s+1) = \theta^n, v\right) = \mathbb{E}\left[G^n/(L^n+G^n)|Y_s^n(k_s+1) = \theta^n, v\right]$. The function $G^n/(L^n+G^n)$ is concave in G^n and convex in L^n. Therefore, using Jensen's inequality and then the Chernoff bound we obtain

$$\mathbb{E}_{G_n}\left[\frac{G_n}{G_n+\bar{L}_n}|Y_s^n(k_s+1) = \theta^n, v\right] \le Q \le \mathbb{E}\left[\frac{\bar{G}_n}{\bar{G}_n+L_n}|Y_s^n(k_s+1) = \theta^n, v\right]$$
$$\frac{(1-\delta)\bar{G}_n}{\bar{G}_n(1-\delta)+\bar{L}_n}(1-e^{-\frac{\delta^2}{2}\bar{G}_n}) \le Q \le \frac{\bar{G}_n}{\bar{G}_n+(1-\delta)\bar{L}_n} + e^{-\frac{\delta^2}{2}\bar{L}_n}.$$

where $Q = \Pr\left(b_p^n\ wins|Y_s^n(k_s+1) = \theta^n, v\right)$. Our assumption $\lim F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|v\right) > 0$ implies either $\bar{G}_n \to \infty$ or $\bar{L}_n \to \infty$ or both. Taking the limits and noting that δ is arbitrary we obtain

$$\lim \Pr\left(b_p^n \; wins | Y_s^n\left(k_s+1\right)=\theta^n, v\right) = \lim \frac{\bar{G}_n}{\bar{G}_n+\bar{L}_n}.$$

Since $\bar{F}_s^n\left(\theta^n|v\right) \in \left(\kappa_s-\epsilon_1, \kappa_s+\epsilon_1\right)$ by assumption, we have

$$\begin{aligned}
\lim \frac{\kappa_s \frac{\left(\kappa_s-\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s+\epsilon_1}}{\kappa_s \frac{\left(\kappa_s+\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s-\epsilon_1}+\left(1-\kappa_s\right)\frac{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)-\kappa_s+\epsilon_1}{1-\kappa_s-\epsilon_1}} &\le \lim \frac{\bar{G}_n}{\bar{G}_n+\bar{L}_n} \\
&\le \lim \frac{\kappa_s \frac{\left(\kappa_s+\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s-\epsilon_1}}{\kappa_s \frac{\left(\kappa_s-\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s+\epsilon_1}+\left(1-\kappa_s\right)\frac{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)-\kappa_s-\epsilon_1}{1-\kappa_s+\epsilon_1}}.
\end{aligned}$$

But $\lim \Pr\left(\bar{F}_s^n\left(Y_s^n\left(k_s+1\right)|v\right) \in \left(\kappa_s-\epsilon_1, \kappa_s+\epsilon_1\right)|v\right)=1$ for every $\epsilon_1>0$ by the LLN. Hence, $\lim \Pr\left(\bar{F}_s^n\left(Y_s^n\left(k_s+1\right)|v\right) \in \left(\kappa_s-\epsilon_1, \kappa_s+\epsilon_1\right)|Y_s^n\left(k_s+1\right) \in \left[\underline{\theta}_p^n, \theta_p^n\right], v\right)=1$. Therefore,

$$\begin{aligned}
\lim \frac{\kappa_s \frac{\left(\kappa_s-\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s+\epsilon_1}}{\kappa_s \frac{\left(\kappa_s+\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s-\epsilon_1}+\left(1-\kappa_s\right)\frac{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)-\kappa_s+\epsilon_1}{1-\kappa_s-\epsilon_1}} &\le \lim \Pr\left(b_p^n \; wins | Y_s^n\left(k_s+1\right) \in \left[\underline{\theta}_p^n, \theta_p^n\right], v\right) \\
&\le \lim \frac{\kappa_s \frac{\left(\kappa_s+\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s-\epsilon_1}}{\kappa_s \frac{\left(\kappa_s-\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s+\epsilon_1}+\left(1-\kappa_s\right)\frac{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)-\kappa_s-\epsilon_1}{1-\kappa_s+\epsilon_1}}.
\end{aligned}$$

Since this is true for each $\epsilon_1>0$, taking $\epsilon_1 \to 0$ shows $\lim \Pr\left(b_p^n \; wins | P^n=b_p^n, v\right) = \lim \frac{\kappa_s-\bar{F}_s^n\left(\theta_p^n|v\right)}{F_s^n\left(\left[\underline{\theta}_p^n,\theta_p^n\right]|v\right)}$. □

The following lemma, which is proven in the online appendix, focuses on the case where the probability that the price is equal to the pooling bid converges to zero. It uses large deviations theory to show that $\Pr\left(b_p^n \; loses | V=v, P^n=b_p^n\right)$ is asymptotically proportional to $\mathbb{E}[L^n|P^n=b_p^n, V=v]/\mathbb{E}[X^n|P^n=b_p^n, V=v]$.

Lemma A.4. *If there is a sequence of pooling bids b_p^n such that $\lim \Pr\left(P^n \ge b_p^n|V=v\right)=0$ and $\lim \sqrt{n}F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|V=v\right)>0$, then*

$$\lim \frac{\Pr\left(b_p^n \; loses | P^n=b_p^n, V=v\right)}{\frac{\overline{F}_s^n\left(\underline{\theta}_p^n|v\right)\left(1-\bar{F}_s\left(\underline{\theta}_p^n|v\right)\right)}{nF_s^n\left(\left[\underline{\theta}_p^n,\theta_p^n\right]|v\right)\left(\kappa_s-\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)\right)}}=1.$$

In the three lemmata below, we use the probabilities of winning at a pooling bid to rule out pooling in certain cases. Recall that $\theta_s(v)=\lim_n \theta_s^n(v)$ and $F_s(\theta|v)=\lim F_s^n\left(\theta|v\right)$ whenever such limits exist.

Lemma A.5. *Fix a sequence of bidding equilibria* **H**. *If $F_s\left(\theta_s(1)|1\right) > F_s\left(\theta_s(0)|1\right)$ or if $F_s\left(1|1\right) \ge F_s\left(1|0\right)$ and $F_s\left(1|1\right)>\kappa_s$, then there is no pooling by pivotal types.*

Proof. We will argue that if $F_s\left(\theta_s(1)|1\right) > F_s\left(\theta_s(0)|1\right)$, then pooling by pivotal types is incompatible with equilibrium. At the end of the proof we show that $F_s\left(1|1\right) \ge F_s\left(1|0\right)$ and $F_s\left(1|1\right)>\kappa_s$ together imply that $F_s\left(\theta_s(1)|1\right) > F_s\left(\theta_s(0)|1\right)$.

The fact that $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$ implies $\theta_s(1) > \theta_s(0)$ and $F_s(\theta_s(1)|0) > F_s(\theta_s(0)|0)$. Pooling by pivotal types implies that $F_s\left(\underline{\theta}_p|v\right) \leq F_s(\theta_s(0)|v) < F_s(\theta_s(1)|v) \leq F_s(\theta_p|v)$. There are three cases to consider: (1) $F_s(\underline{\theta}_p|v) < F_s(\theta_s(0)|v)$ and $F_s(\theta_p|v) > F_s(\theta_s(1)|v)$; (2) $F_s(\theta_p|v) = F_s(\theta_s(1)|v)$ and $F_s\left(\underline{\theta}_p|v\right) < F_s(\theta_s(0)|v)$; and (3) $F_s\left(\underline{\theta}_p|v\right) = F_s(\theta_s(0)|v)$. In each of these cases, we will show that pooling by pivotal types is incompatible with equilibrium behavior.

Case 1: $F_s(\underline{\theta}_p|v) < F_s(\theta_s(0)|v)$ and $F_s(\theta_p|v) > F_s(\theta_s(1)|v)$. In this case,

$$\lim \Pr\left(P^n = b_p^n, b_p^n \quad loses|V = v\right) = \frac{\kappa_s - \bar{F}_s(\theta_p|v)}{F_s\left(\left[\underline{\theta}_p, \theta_p\right]|v\right)} = \frac{F_s(\theta_s(v)|1) - F_s\left(\underline{\theta}_p|v\right)}{F_s(\theta_p|v) - F_s(\underline{\theta}_p|v)}, \tag{A.3}$$

by Lemma A.3. In order for type θ_p to be willing to bid the pooling bid b instead of bidding slightly above the pooling bid, we need the following incentive-compatibility inequality to be satisfied:

$$(1-b_p)\, l(\theta_p)\left(1 - \frac{F_s(\theta_s(1)|1) - F_s\left(\underline{\theta}_p|1\right)}{F_s(\theta_p|1) - F_s\left(\underline{\theta}_p|1\right)}\right) - b_p\left(1 - \frac{F_s(\theta_s(0)|0) - F_s\left(\underline{\theta}_p|0\right)}{F_s(\theta_p|0) - F_s\left(\underline{\theta}_p|0\right)}\right) \geq (1-b_p)\, l(\theta_p) - b_p.$$

Therefore

$$\frac{b_p}{1-b_p} \geq l(\theta_p)\left(\frac{F_s(\theta_s(1)|1) - F_s\left(\underline{\theta}_p|1\right)}{F_s(\theta_s(0)|0) - F_s\left(\underline{\theta}_p|0\right)}\right)\left(\frac{F_s(\theta_p|0) - F_s\left(\underline{\theta}_p|0\right)}{F_s(\theta_p|1) - F_s\left(\underline{\theta}_p|1\right)}\right).$$

In order for pooling to be individually rational for type $\underline{\theta}_p$, we need the following individual rational inequality to be satisfied:

$$(1-b_p)\, l\left(\underline{\theta}_p\right)\left(1 - \frac{F_s(\theta_s(1)|1) - F_s\left(\underline{\theta}_p|1\right)}{F_s(\theta_p|1) - F_s\left(\underline{\theta}_p|1\right)}\right) - l\left(\underline{\theta}_p\right) b_p\left(1 - \frac{F_s(\theta_s(0)|0) - F_s\left(\underline{\theta}_p|0\right)}{F_s(\theta_p|0) - F_s\left(\underline{\theta}_p|0\right)}\right) \geq 0.$$

Therefore,

$$\frac{b_p}{1-b_p} \leq l\left(\underline{\theta}_p\right)\left(\frac{F_s(\theta_p|1) - F_s(\theta_s(1)|1)}{F_m(\theta_p|0) - F_s(\theta_s(0)|0)}\right)\left(\frac{F_s(\theta_p|0) - F_s\left(\underline{\theta}_p|0\right)}{F_s(\theta_p|1) - F_s\left(\underline{\theta}_p|1\right)}\right).$$

Combining the incentive compatibility and individual rationality inequalities, we obtain

$$l\left(\underline{\theta}_p\right)\left(\frac{F_s(\theta_p|1) - F_s(\theta_s(1)|1)}{F_s(\theta_p|0) - F_s(\theta_s(0)|0)}\right) \geq l(\theta_p)\left(\frac{F_s(\theta_s(1)|1) - F_s\left(\underline{\theta}_p|1\right)}{F_s(\theta_s(0)|0) - F_s\left(\underline{\theta}_p|0\right)}\right),$$

which is not possible because

$$l\left(\underline{\theta}_p\right) \leq \frac{F_s(\theta_s(0)|1) - F_s\left(\underline{\theta}_p|1\right)}{F_s(\theta_s(0)|0) - F_s\left(\underline{\theta}_p|0\right)} < \frac{F_s(\theta_s(1)|1) - F_s\left(\underline{\theta}_p|1\right)}{F_s(\theta_s(0)|0) - F_s\left(\underline{\theta}_p|0\right)}$$

and because

$$\frac{F_s(\theta_p|1) - F_s(\theta_s(1)|1)}{F_s(\theta_p|0) - F_s(\theta_s(0)|0)} < \frac{F_s(\theta_p|1) - F_s(\theta_s(1)|1)}{F_s(\theta_p|0) - F_s(\theta_s(1)|0)} \leq l(\theta_p)$$

by MLRP.

Case 2: If $F_s(\theta_p|v) = F_s(\theta_s(1)|v)$ and $F_s(\underline{\theta}_p|v) < F_s(\theta_s(0)|v)$, then Lemma A.3 implies that $\lim \Pr(P^n = b_p^n, b_p^n\ wins|V = 1) = 0$ and $\lim \Pr(P^n = b_p^n, b_p^n\ wins|V = 0) > 0$, showing that Inequality (A.2) is satisfied.

Case 3: If $F_s(\underline{\theta}_p|v) = F_s(\theta_s(0)|v)$, then Lemma A.3 implies that $\lim \Pr(P^n = b_p^n, b_p^n\ wins|V = 0) = 1$ and $\lim \Pr(P^n = b_p^n, b_p^n\ wins|V = 1) < 1$ again showing that Inequality (A.2) is satisfied.

We conclude the proof by arguing that $F_s(1|1) \geq F_s(1|0)$ and $F_s(1|1) > \kappa_s$ together imply that $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$. On the way to a contradiction assume $F_s(\theta_s(1)|1) \leq F_s(\theta_s(0)|1)$. Note $F_s(1|1) > \kappa_s$ implies $0 < F_s(\theta_s(1)|1) \leq F_s(\theta_s(0)|1)$. Our assumption $F_s(\theta_s(1)|1) \leq F_s(\theta_s(0)|1)$ and MLRP together imply that $1 \geq \bar{F}_s(\theta_s(1)|1)/\bar{F}_s(\theta_s(1)|0) > F_s(\theta_s(1)|1)/F_s(\theta_s(1)|0)$. However, $F_s(1|V = v) = \bar{F}_s(\theta_s(1)|V = v) + F_s(\theta_s(1)|V = v)$, $\bar{F}_s(\theta_s(1)|1) \leq \bar{F}_s(\theta_s(1)|0)$, and $F_s(\theta_s(1)|1) < F_s(\theta_s(1)|0)$ together imply that $F_s(1|1) < F_s(1|0)$ leading to a contradiction. □

However, the previous lemma did not address the case where $\lim \bar{F}_s^n(\theta_s^n(1)|v) = \lim \bar{F}_s^n(\theta_s^n(0)|v)$. In the following lemma we further show that if the pivotal types are distinct and if there is pooling by pivotal types, then $\lim \Pr(P^n \leq b_p^n|V = 1) = 1$ and $\lim \Pr(P^n < b_p^n|V = 0) = 0$. The lemma implies that if there is pooling by pivotal types and the pivotal types are distinct, then $\bar{F}_s^n(\theta_s^n(1)|v) > \bar{F}_s^n(\theta_s^n(0)|v)$ for all n sufficiently large, i.e., the order of the pivotal types is reversed.

Lemma A.6. *Fix a sequence of bidding equilibria* $\mathbf{H}$ *and suppose that* $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|V = v) - \bar{F}_s^n(\theta_s^n(0)|V = v)| \to \infty$. *If there is pooling by pivotal types, then* $\lim \Pr(P^n \leq b_p^n|V = 1) = 1$ *and* $\lim \Pr(P^n < b_p^n|V = 0) = 0$.

Proof. Pooling by pivotal types implies that $\lim \Pr\left(P^n = b_p^n|V = v\right) > 0$ for $v = 0, 1$. Suppose $\lim \Pr(P^n < b_p^n|V = 0) > 0$ then $\lim \sqrt{n}\left(F_s^n(\theta_s(0)|0) - F_s^n\left(\underline{\theta}_p^n|0\right)\right) \in (-\infty, \infty)$. Moreover, $\lim \Pr\left(P^n = b_p^n|V = 1\right) > 0$ and $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|V = 1) - \bar{F}_s^n(\theta_s^n(0)|V = 1| \to \infty$ together imply $\lim \sqrt{n}\left(F_s^n(\theta_s(1)|1) - F_s^n\left(\underline{\theta}_p^n|1\right)\right) = \infty$. Along any sequence where the limit in the equation below exists, Lemma A.3 implies that there is a constant C such that

$$\begin{aligned}\lim \frac{\Pr\left(b^n\ loses|P^n = b_p^n, V = 0\right)}{\Pr\left(b^n\ loses|P^n = b_p^n, V = 1\right)} &= C \lim \frac{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|1\right)}{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|0\right)} \frac{\frac{1}{\sqrt{n}}}{\left(F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\underline{\theta}_p^n|1\right)\right)} \\ &\leq \frac{1}{\eta} \frac{C}{\sqrt{n}\left(F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\underline{\theta}_p^n|1\right)\right)} = 0\end{aligned}$$

showing that pooling is not possible. Therefore, if there is pooling by pivotal types, then $\lim \Pr(P^n < b_p^n|V = 0) = 0$.

Suppose $\lim \Pr(P^n \leq b_p^n|V = 1) < 1$. Then $\lim \sqrt{n}\left(F_s^n\left(\theta_p^n|1\right) - F_s^n\left(\theta_s^n(1)|1\right)\right) \in (-\infty, \infty)$. Moreover, $\lim \Pr\left(P^n = b_p^n|V = 0\right) > 0$ and $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|V = 1) - \bar{F}_s^n(\theta_s^n(0)|V = 1| \to \infty$ together imply $\lim \sqrt{n}\left(F_s^n\left(\theta_p^n|0\right) - F_s^n\left(\theta_s^n(0)|0\right)\right) = \infty$. Using Lemma A.3 we obtain

$$\lim \frac{\Pr\left(b^n\ wins|P^n = b_p^n, V = 1\right)}{\Pr\left(b^n\ wins|P^n = b_p^n, V = 0\right)} = C \lim \frac{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|0\right)}{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|1\right)} \frac{\frac{1}{\sqrt{n}}}{\left(F_s^n\left(\theta_p^n|0\right) - F_s^n\left(\theta_s^n(0)|0\right)\right)} = 0$$

again showing that pooling is not possible. Therefore, if there is pooling by pivotal types, then $\lim \Pr(P^n \le b_p^n | V = 1) = 1$. $\square$

The following lemma shows that there cannot be a pool that occurs with positive probability in state $V = 1$ and probability zero in state $V = 0$ if the pivotal types are distinct.

Lemma A.7. *Fix a sequence of bidding equilibria* $\mathbf{H}$ *and assume* $\sqrt{n}\left(F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\theta_s^n(0)|1\right)\right) \to \infty$. *There is no sequence of pooling bids* b_p^n *such that*

$$\lim \Pr\left(P^n = b_p^n | V = 1\right) > 0 \text{ and } \lim \Pr\left(P^n \ge b_p^n | V = 0\right) = 0.$$

Proof. We will show that

$$\lim \frac{\Pr\left(b^n\ loses | P^n = b_p^n, V = 0\right)}{\Pr\left(b^n\ loses | P^n = b_p^n, V = 1\right)} = 0$$

however this implies that there is loser's curse at the pooling bid for sufficiently large n and therefore pooling cannot be sustained. Lemma A.4 and Lemma A.3 together imply

$$\lim \frac{\Pr\left(b^n\ loses | P^n = b_p^n, V = 0\right)}{\Pr\left(b^n\ loses | P^n = b_p^n, V = 1\right)} =$$

$$C \lim \frac{\frac{\overline{F}_s^n\left(\underline{\theta}_p^n|0\right)\left(1-\bar{F}_s\left(\underline{\theta}_p^n|0\right)\right)}{nF_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|0\right)\left(\kappa_s - \bar{F}_s^n\left(\underline{\theta}_p^n|0\right)\right)}}{\frac{\max\left\{F_s^n\left(\theta_s(1)|1\right) - F_s^n\left(\underline{\theta}_p^n|1\right), \frac{1}{\sqrt{n}}\right\}}{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|1\right)}} =$$

$$\lim \frac{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|1\right)}{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|0\right)} \frac{C\overline{F}_s^n\left(\underline{\theta}_p^n|0\right)\left(1 - \bar{F}_s\left(\underline{\theta}_p^n|0\right)\right)}{n\left(\kappa_s - \bar{F}_s^n\left(\underline{\theta}_p^n|0\right)\right)\max\left\{F_s^n\left(\theta_s(1)|1\right) - F_s^n\left(\underline{\theta}_p^n|1\right), \frac{1}{\sqrt{n}}\right\}}$$

where $C \in (0, \infty)$. However, $\frac{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|1\right)}{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|0\right)} \le \frac{1}{\eta}$ by Lemma A.8, $n \max\left\{F_s^n\left(\theta_s(1)|1\right) - F_s^n\left(\underline{\theta}_p^n|1\right), \frac{1}{\sqrt{n}}\right\} \ge \sqrt{n}$, $\overline{F}_s^n\left(\underline{\theta}_p^n|0\right)\left(1 - \bar{F}_s\left(\underline{\theta}_p^n|0\right)\right) \le 1$ and $\lim \sqrt{n}\left(\kappa_s - \bar{F}_s^n\left(\underline{\theta}_p^n|0\right)\right) = \infty$ because $\lim \Pr\left(P^n \ge b_p^n | V = 0\right) = 0$. Therefore,

$$\lim \frac{\Pr\left(b^n\ loses | P^n = b_p^n, V = 0\right)}{\Pr\left(b^n\ loses | P^n = b_p^n, V = 1\right)} \le \lim \frac{1}{\eta} \frac{1}{C\sqrt{n}\left(\kappa_s - \bar{F}_s^n\left(\underline{\theta}_p^n|0\right)\right)} = 0.$$

$\square$

A.2. Information content of being pivotal. In this subsection, we provide bounds for the ratio $\frac{\Pr(Y^n(k_s+1)=\theta^n|V=1)}{\Pr(Y^n(k_s+1)=\theta^n|V=0)}$, i.e., the information content of the event of being pivotal. We will then use these bounds to show that distinct pivotal types are a necessary condition for information aggregation. Also, we will use the results in this subsection together with the results in the previous subsection to argue that distinct pivotal types and no pooling by pivotal types together imply information aggregation (Lemma 2.2). The results we present below show that the event of being pivotal provides only bounded amounts of information for the types that set the price if the pivotal types are arbitrarily close.

We begin with the following lemma that outlines the implication of our assumption that there are no arbitrarily informative signals, i.e., $l(\theta) \in (\eta, 1/\eta)$ for each $\theta \in [0,1]$.

Lemma A.8. *For any interval $I \subset [0,1]$, $F_s^n(I|V=1) \in \left[\eta F_s^n(I|V=0), \frac{F_s^n(I|V=0)}{\eta}\right]$. Therefore, $\sqrt{n}\,|F_s^n(\theta_s^n(1)|0) - F_s^n(\theta_s^n(0)|0)| < \infty$ if and only if $\sqrt{n}\,|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| < \infty$.*

Proof. To see this, note $F_s^n(I|V=1) = \int_I a(\theta) f(\theta|1) d\theta = \int_I a(\theta) f(\theta|0) l(\theta) d\theta$ and

$$\eta F_s^n(I|V=0) = \eta \int_I a(\theta) f(\theta|0) d\theta \leq \int_I a(\theta) f(\theta|0) l(\theta) d\theta \leq \frac{1}{\eta}\int_I a(\theta) f(\theta|0) d\theta = \frac{1}{\eta} F_s^n(I|V=0)$$

because $l(\theta) \in (\eta, 1/\eta)$ for $\theta \in [0,1]$. □

For any $\theta \in [0,1]$ and $v = 0,1$ define

$$\begin{aligned} z_v^n(\theta) &:= \frac{k_s - (n-1)\bar{F}_s^n(\theta|v)}{\sqrt{\left(1-\bar{F}_s^n(\theta|v)\right)\bar{F}_s^n(\theta|v)(n-1)}} \\ &= \sqrt{\frac{(n-1)}{\left(1-\bar{F}_s^n(\theta|v)\right)\bar{F}_s^n(\theta|v)}}\left(\frac{k_s}{n-1} - \bar{F}_s^n(\theta|v)\right) \end{aligned}$$

Intuitively, $z_v^n(\theta)$ measures the distance between type θ and the pivotal type $\theta_s^n(v)$ in terms of standard deviations.

The probability that a particular type θ is pivotal (i.e., $Y_s^n(k_s+1) = \theta$) can be approximated using the central limit theorem. If $\lim \frac{n\kappa_s - n\bar{F}_s^n(\theta|v)}{\sqrt{n\bar{F}_s^n(\theta|v)(1-\bar{F}_s^n(\theta|v))}} = a$, then

$$Bi(k_s; n, \bar{F}_s^n(\theta|v)) \to \Phi(a)$$

where Φ denotes the standard normal cumulative distribution. Moreover, if we let $p = \bar{F}_s^n(\theta|v)$, then

$$bi(k_s; n, p) = \binom{n}{k_s} p^{k_s}(1-p)^{n-k_s} = \frac{1+\delta_n(p)}{\sqrt{2\pi p(1-p)n}}\phi\left(\frac{k_s - np}{\sqrt{p(1-p)n}}\right) \tag{A.4}$$

where ϕ denotes the standard normal density, and $\lim_{n\to\infty}\sup_{p:|np-k_s|<n^t}\delta_n(p) = 0$ for $t < 2/3$, i.e., the binomial density converges to the normal density uniformly over the set of p and k_s such that $|np - k_s| < n^t$ for $t < \frac{2}{3}$ (see Lesigne (2005, Proposition 8.2)). In the following two lemmata, we use these convergence results and show that if the price is set by a type θ that is within finitely many standard deviations of both pivotal types, then the information that this type gets from being pivotal is bounded.

Lemma A.9. *Pick a sequence of types $\{\theta^n\}$ that bid in market s. Assume that $\lim z_v^n(\theta^n) = z_v$ for $v = 0,1$ and $\lim l(\theta^n) = \rho$. For any $\delta > 0$, there exists an N such that for all $n > N$ we have*

$$(1-\delta)\frac{\phi(z_1)}{\phi(z_0)}\rho \leq \frac{\Pr(Y^n(k_s+1) = \theta^n|V=1)}{\Pr(Y^n(k_s+1) = \theta^n|V=0)} \leq \rho(1+\delta)\frac{\phi(z_1)}{\phi(z_0)}.$$

Therefore, $\frac{\Pr(Y^n(k_s+1)=\theta^n|V=1)}{\Pr(Y^n(k_s+1)=\theta^n|V=0)} \to \frac{\phi(z_1)}{\phi(z_0)}\rho$.

Proof. A direct computation shows that

$$\frac{\Pr\left(Y^n(k_s+1)=\theta^n|V=1\right)}{\Pr\left(Y^n(k_s+1)=\theta^n|V=0\right)}=l(\theta^n)\frac{bi\left(k_s;n-1,\bar{F}_s^n(\theta^n|1)\right)}{bi\left(k_s;n-1,\bar{F}_s^n(\theta^n|0)\right)}$$

where $bi(k;n,r)$ is the binomial density function with probability of success equal to r. Equation (A.4) implies that for any $\delta>0$, there exists an N such that

$$(1-\delta)\frac{\phi\left(z_1^n(\theta^n)\right)}{\phi\left(z_0^n(\theta^n)\right)}\sqrt{\frac{\left(1-\bar{F}_s^n(\theta^n|0)\right)\bar{F}_s^n(\theta^n|0)}{\left(1-\bar{F}_s^n(\theta^n|1)\right)\bar{F}_s^n(\theta^n|1)}}\leq\frac{bi\left(k_s;n-1,\bar{F}_s^n(\theta^n|1)\right)}{bi\left(k_s;n-1,\bar{F}_s^n(\theta^n|0)\right)}\leq$$
$$(1+\delta)\frac{\phi\left(z_1^n(\theta^n)\right)}{\phi\left(z_0^n(\theta^n)\right)}\sqrt{\frac{\left(1-\bar{F}_s^n(\theta^n|0)\right)\bar{F}_s^n(\theta^n|0)}{\left(1-\bar{F}_s^n(\theta^n|1)\right)\bar{F}_s^n(\theta^n|1)}}$$

for all $n>N$. Our assumption that $\lim z_v^n(\theta^n)=z_v$ and $\frac{k_s}{n-1}\to\kappa_s$ together establish that $\lim\sqrt{n}\left|\bar{F}_s^n(\theta^n|v)-\kappa_s\right|<\infty$ for $v=0,1$. Also, $\lim\sqrt{n}\left|\bar{F}_s^n(\theta^n|v)-\kappa_s\right|<\infty$ implies

$$\sqrt{\frac{\left(1-\bar{F}_s^n(\theta^n|0)\right)\bar{F}_s^n(\theta^n|0)}{\left(1-\bar{F}_s^n(\theta^n|1)\right)\bar{F}_s^n(\theta^n|1)}}\to 1.$$

The fact that $\phi\left(z_v^n(\theta)\right)$ and $\sqrt{\frac{\left(1-\bar{F}_s^n(\theta^n|0)\right)\bar{F}_s^n(\theta^n|0)}{\left(1-\bar{F}_s^n(\theta^n|1)\right)\bar{F}_s^n(\theta^n|1)}}$ are continuous functions of θ imply that for any $\delta>0$, there exists an N such that for all $n>N$ we have

$$(1-\delta)\frac{\phi\left(z_1\right)}{\phi\left(z_0\right)}\rho\leq\frac{\Pr\left(Y^n(k_s+1)=\theta^n|V=1\right)}{\Pr\left(Y^n(k_s+1)=\theta^n|V=0\right)}\leq(1+\delta)\frac{\phi\left(z_1\right)}{\phi\left(z_0\right)}\rho.$$

□

Lemma A.10. *Assume* $\lim\sqrt{n}\left|F_s^n\left(\theta_s^n(1)|1\right)-F_s^n\left(\theta_s^n(0)|1\right)\right|=x<\infty$ *and* $\lim\sqrt{n}\left(\kappa_s-\bar{F}_s^n\left(0|0\right)\right)=-\infty$. *Suppose* θ_y^n *is a the type such that* $F_s^n\left(\left[\theta_y^n,\theta_s^n(0)\right]|0\right)=\sqrt{\kappa_s(1-\kappa_s)}y/\sqrt{n}$. *For any* $\delta>0$, *there exists an* N *such that for all* $n>N$ *and for any interval* $[a,b]\subset\left[\theta_y^n,\theta_s^n(0)\right]$ *such that* $F_s^n\left([a,b]|0\right)>0$ *we have*

$$(1-\delta)\frac{\phi\left(x+\frac{y}{\eta}\right)}{\phi\left(0\right)}\eta\leq\frac{\Pr\left(Y^n(k_s+1)\in[a,b]|V=1\right)}{\Pr\left(Y^n(k_s+1)\in[a,b]|V=0\right)}\leq(1+\delta)\frac{\phi\left(0\right)}{\phi\left(y\right)}\frac{1}{\eta}.$$

Proof. Suppose, without loss of generality, that $\lim\frac{\sqrt{n}}{\sqrt{\kappa_s(1-\kappa_s)}}\left(F_s^n\left(\theta_s^n(1)|1\right)-F_s^n\left(\theta_s^n(0)|1\right)\right)\geq 0$. Note that if $\lim\sqrt{n}\left(\kappa_s-\bar{F}_s^n\left(0|0\right)\right)=-\infty$, then $\lim\sqrt{n}F_s(\theta_s^n(0)|0)=\infty$ and the interval $\left[\theta_y^n,\theta_s^n(0)\right]$ is well defined for all sufficiently large n. For any sequence $\{\theta^n\}$ such that $\theta^n\in\left[\theta_y^n,\theta_s^n(0)\right]$ for every n, we have $\lim z_v^n(\theta^n)=\lim\frac{\sqrt{n}\left(\kappa_s-\bar{F}_s^n(\theta^n|v)\right)}{\sqrt{\kappa_s(1-\kappa_s)}}$. Also, $l(\theta)\in[\eta,\frac{1}{\eta}]$ (because there are no arbitrarily informative signals), $\lim z_1^n(\theta^n)\in\left[-x-\frac{y}{\eta},0\right]$, and $\lim z_0^n(\theta^n)\in[-y,0]$. Therefore, Lemma A.9 implies that for any $\delta>0$, there exists an N such that for all $n>N$ and any $\theta \in \left[\theta_y^n, \theta_s^n(0)\right]$

$$(1-\delta)\frac{\phi\left(x+\frac{y}{\eta}\right)}{\phi(0)} \leq \frac{bi\left(k_s; n-1, \bar{F}_s^n(\theta|1)\right)}{bi\left(k_s; n-1, \bar{F}_s^n(\theta|0)\right)} \leq (1+\delta)\frac{\phi(0)}{\phi(y)}.$$

Thus using the fact that $l(\theta) \in [\eta, \frac{1}{\eta}]$, we conclude that

$$(1-\delta)\frac{\phi\left(x+\frac{y}{\eta}\right)}{\phi(0)}\eta \leq \frac{\Pr\left(Y^n(k_s+1) \in [a,b]|V=1\right)}{\Pr\left(Y^n(k_s+1) \in [a,b]|V=0\right)} \leq (1+\delta)\frac{\phi(0)}{\phi(y)}\frac{1}{\eta}$$

for any interval $[a,b] \subset \left[\theta_y^n, \theta^n(0)\right]$. □

A.3. Proof of Information Aggregation Lemmata 2.2 and 2.3.

Proof of Lemma 2.2. First we argue that if **H** aggregates information, then there is no pooling by pivotal types and the pivotal types are distinct. Note that if there is pooling by pivotal types, then **H** does not aggregate information by definition.[30] Let $\sigma := \sqrt{\kappa_s(1-\kappa_s)}$ and recall that $\bar{F}_s^n(0|0)$ is the fraction of types who bid in market s in state 0. We will argue that if **H** aggregates information, then the pivotal types are distinct, i.e.,

$$\lim \sqrt{n}\left|F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\theta_s^n(0)|1\right)\right| = \infty.$$

Suppose the pivotal types are arbitrarily close, i.e.,

$$\lim \sqrt{n}\left|F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\theta_s^n(0)|1\right)\right| < \infty.$$

In the next two claims, we will show 1) If the number of objects exceeds the number of bidders with positive probability in state $V=0$ (i.e., if $\lim \sqrt{n}\left(\kappa_s - \bar{F}_s^n(0|0)\right) > -\infty$), then **H** does not aggregate information; and 2) If the number of bidders exceeds the number of objects with probability one in state $V=0$ (i.e., if $\lim \sqrt{n}\left(\kappa_s - \bar{F}_s^n(0|0)\right) = -\infty$), then **H** does not aggregate information. Therefore, we will conclude that if the pivotal types are arbitrarily close, then **H** does not aggregate information establishing our claim.

Claim A.1. If $\lim \sqrt{n}\left|F_s^n\left(\theta_s^n(1)|V=1\right) - F_s^n\left(\theta_s^n(0)|V=1\right)\right| < \infty$ and $\lim \sqrt{n}\left(\kappa_s - \bar{F}_s^n(0|0)\right) > -\infty$, then **H** does not aggregate information.

Proof. Suppose $\sqrt{n}\left|F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\theta_s^n(0)|1\right)\right| < \infty$. We will show that the price is equal to zero with strictly positive probability in both states and therefore **H** does not aggregate information. Suppose that $\frac{\sqrt{n}}{\sigma}\left(\kappa_s - \bar{F}_s^n(0|0)\right) \to x > -\infty$ where x is possibly equal to $+\infty$. The central limit theorem implies that the number of goods in the auction exceeds the number of bidders with positive probability if $V=0$, and $\lim_n \Pr\left(Y_s^n(k_s+1) = 0|V=0\right) = \Phi(x) > 0$.

Below we argue that $\sqrt{n}\left|F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\theta_s^n(0)|1\right)\right| < \infty$ and $\frac{\sqrt{n}}{\sigma}\left(\kappa_s - \bar{F}_s^n(0|0)\right) \to x > -\infty$ together imply that $\frac{\sqrt{n}}{\sigma}\left(\kappa_s - \bar{F}_s^n(0|1)\right) \to x' > -\infty$. But if $\frac{\sqrt{n}}{\sigma}\left(\kappa_s - \bar{F}_s^n(0|1)\right) \to x' > -\infty$, then applying the central limit theorem once again we find $\lim_n \Pr\left(Y_s^n(k_s+1) = 0|V=1\right) = \Phi(x') >$

[30] This is because $\lim \Pr\left(P^n = b_p^n|V=v\right) > 0$ for $v = 0, 1$, i.e., the auction price is equal to the pooling bid with strictly positive probability in both states.

0 and therefore $\lim_n \Pr(P^n = 0|V = 1) \geq \Phi(x') > 0$. However, $\lim_n \Pr(P^n = 0|V = v) > 0$ for $v = 0, 1$ and $\lim_n \frac{\Pr(V=1|P^n=0)}{\Pr(V=0|P^n=0)} = \frac{\Phi(x')}{\Phi(x)} \in (0, \infty)$ contradicts that $\mathbf{H}$ aggregates information.

We argue that $\frac{\sqrt{n}}{\sigma}(\kappa_s - \bar{F}_s^n(0|0)) = \frac{\sqrt{n}}{\sigma}\left(\kappa_s - \left(F_s^n(\theta_s^n(0)|0) + \bar{F}_s^n(\theta_s^n(0)|0)\right)\right) \to x > -\infty$ implies $\lim \sqrt{n} F_s(\theta_s^n(0)|0) < \infty$ and therefore $\lim \sqrt{n} F_s(\theta_s^n(0)|1) < \infty$. By definition we have $\bar{F}_s^n(\theta_s^n(0)|0) = \kappa_s$ if $\kappa_s \leq \bar{F}_s^n(0|0)$ and $\theta_s^n(0) = 0$ otherwise. Therefore, $\sqrt{n}(\kappa_s - \bar{F}_s^n(0|0)) = -\infty$ if and only if $\lim \sqrt{n} F_s(\theta_s^n(0)|0) = \infty$. Hence, $\frac{\sqrt{n}}{\sigma}(\kappa_s - \bar{F}_s^n(0|0)) \to x > -\infty$ implies that $\lim \sqrt{n} F_s(\theta_s^n(0)|0) < \infty$ and hence $\lim \sqrt{n} F_s(\theta_s^n(0)|1) < \infty$ by Lemma A.8.

We now show that $\lim \sqrt{n}\left(\kappa_s - \bar{F}_s^n(0|0)\right) = x > -\infty$ implies $\lim \sqrt{n}\left(\kappa_s - \bar{F}_s^n(0|1)\right) > -\infty$. We argued in the previous paragraph that $\frac{\sqrt{n}}{\sigma}(\kappa_s - \bar{F}_s^n(0|1)) \to -\infty$ if and only if $\lim \sqrt{n} F_s(\theta_s^n(1)|1) = \infty$. However, if $\lim \sqrt{n} F_s(\theta_s^n(1)|1) = \infty$, then $\lim \sqrt{n} F_s^n([\theta_s^n(0), \theta_s^n(1)]|1) = \infty$ because $\lim \sqrt{n} F_s(\theta_s^n(0)|0) < \infty$ and because $\lim \sqrt{n} F_s(\theta_s^n(0)|1) < \infty$. But this contradicts $\sqrt{n}\left|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)\right| < \infty$. Hence, $\frac{\sqrt{n}}{\sigma}(\kappa_s - \bar{F}_s^n(0|1)) \to x'$ for some $x' > -\infty$ which is possibly equal to $+\infty$. □

We now turn to the case where $\lim \sqrt{n}\left(\kappa_s - \bar{F}_s^n(0|0)\right) = -\infty$. Pick any $y > 0$ and let θ_y^n denote the type such that

$$F_s^n\left(\left[\theta_y^n, \theta_s^n(0)\right]|0\right) = \sigma y/\sqrt{n} \tag{A.5}$$

when such a type exists. Observe that $\theta_y^n < \theta_{\frac{2}{3}y}^n < \theta_{\frac{1}{3}y}^n < \theta^n(0)$ and $F_s^n\left(\left[\theta_{\frac{2}{3}y}^n, \theta_{\frac{1}{3}y}^n\right]|0\right) = \sigma y/3/\sqrt{n}$ by the definition of these types given in equation (A.5). Let

$$A^n := \left\{p : p = b^n(\theta), \theta \in \left[\theta_{\frac{2}{3}y}^n, \theta_{\frac{1}{3}y}^n\right]\right\}.$$

The central limit theorem implies that

$$\lim \Pr\left(Y^n(k+1) \in \left[\theta_{\frac{2}{3}y}^n, \theta_{\frac{1}{3}y}^n\right]|V = 0\right) = \Phi\left(\frac{2}{3}y\right) - \Phi\left(\frac{1}{3}y\right) > 0.$$

Also,

$$\Pr(P^n \in A^n|V = 0) \geq \Pr\left(Y^n(k+1) \in \left[\theta_{\frac{2}{3}y}^n, \theta_{\frac{1}{3}y}^n\right]|V = 0\right)$$

because $P^n = b^n(Y^n(k+1))$. The inequality above does not necessarily hold as an equality because types other than those $\left[\theta_{\frac{2}{3}y}^n, \theta_{\frac{1}{3}y}^n\right]$ may also choose a bid in A^n. Lemma A.10 implies that

$$\lim \Pr\left(Y^n(k+1) \in \left[\theta_{\frac{2}{3}y}^n, \theta_{\frac{1}{3}y}^n\right]|V = 1\right) \geq \frac{\phi\left(x + \frac{y}{\eta}\right)}{\phi(0)}\eta\left(\Phi\left(\frac{2}{3}y\right) - \Phi\left(\frac{1}{3}y\right)\right) > 0.$$

and therefore

$$\lim \Pr(P^n \in A^n|V = 1) \geq \frac{\phi\left(x + \frac{y}{\eta}\right)}{\phi(0)}\eta\left(\Phi\left(\frac{2}{3}y\right) - \Phi\left(\frac{1}{3}y\right)\right) > 0.$$

Claim A.2. If $\lim \sqrt{n}\left|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)\right| = x < \infty$ and $\lim \sqrt{n}\left(\kappa_s - \bar{F}_s^n(0|0)\right) = -\infty$, then $\mathbf{H}$ does not aggregate information.

Proof. We will argue that there exists an $\epsilon > 0$ such that $\frac{\Pr(V=1|P^n=p)}{\Pr(V=0|P^n=p)} \in (\epsilon, \frac{1}{\epsilon})$ for any $p \in A^n$

and any n sufficiently large. However, this together with the facts that $\Pr(P^n \in A^n | V=1) > 0$ and $\Pr(P^n \in A^n | V=0) > 0$ imply that $\mathbf{H}$ does not aggregate information.

Pick any $\delta > 0$. For any $\theta^* \in \left[\theta^n_{\frac{2}{3}y}, \theta^n_{\frac{1}{3}y}\right]$ that bids in market s with positive probability, we have either 1) $\{\theta : b^n(\theta) = b^n(\theta^*)\} \subset \left[\theta^n_y, \theta^n(0)\right]$ or 2) $\{\theta : b^n(\theta) = b^n(\theta^*)\} \not\subseteq \left[\theta^n_y, \theta^n(0)\right]$. Moreover, the fact that the bidding function is monotone implies that the set $\{\theta : b^n(\theta) = b^n(\theta^*)\}$ is either a singleton or an interval.

If $\{\theta : b^n(\theta) = b^n(\theta^*)\} \subset \left[\theta^n_y, \theta^n(0)\right]$, then Lemma A.10 implies that

$$(1-\delta)\frac{\phi\left(x+\frac{y}{\eta}\right)}{\phi(0)}\eta \le \frac{\Pr\left(Y^n(k+1) \in \{\theta : b^n(\theta) = b^n(\theta^*)\}|V=1\right)}{\Pr\left(Y^n(k+1) \in \{\theta : b^n(\theta) = b^n(\theta^*)\}|V=0\right)} \le (1+\delta)\frac{\phi(0)}{\phi(y)}\frac{1}{\eta}$$

for all $n > N(\delta)$.[31] Therefore,

$$(1-\delta)\frac{\phi\left(x+\frac{y}{\eta}\right)}{\phi(0)}\eta \le \frac{\Pr\left(P^n = b^n(\theta^*)|V=1\right)}{\Pr\left(P^n = b^n(\theta^*)|V=0\right)} = \frac{\Pr\left(V=1|P^n = b^n(\theta^*)\right)}{\Pr\left(V=0|P^n = b^n(\theta^*)\right)} \le (1+\delta)\frac{\phi(0)}{\phi(y)}\frac{1}{\eta}$$

for all $n > N(\delta)$.

If, on the other hand, $\{\theta : b^n(\theta) = b^n(\theta^*)\} \not\subseteq \left[\theta^n_y, \theta^n(0)\right]$, then either $\left[\theta^n_y, \theta^n_{\frac{2}{3}y}\right] \subset \{\theta : b^n(\theta) = b^n(\theta^*)\}$ or $\left[\theta^n_{\frac{1}{3}y}, \theta^n(0)\right] \subset \{\theta : b^n(\theta) = b^n(\theta^*)\}$ because the set $\{\theta : b^n(\theta) = b^n(\theta^*)\}$ is an interval that extends beyond $\left[\theta^n_y, \theta^n(0)\right]$. Therefore, $\Pr\left(P^n = b^n(\theta^*)|V=v\right) \ge \Pr\left(Y^n(k+1) \in \left[\theta^n_{\frac{1}{3}y}, \theta^n(0)\right]|V=v\right)$ or $\Pr\left(P^n = b^n(\theta^*)|V=v\right) \ge \Pr\left(Y^n(k+1) \in \left[\theta^n_y, \theta^n_{\frac{2}{3}y}\right]|V=v\right)$. The central limit theorem implies that

$$(1-\delta)\left(\Phi\left(\frac{y}{3}\right)-\frac{1}{2}\right) \le \Pr\left(Y^n(k+1) \in \left[\theta^n_{\frac{1}{3}y}, \theta^n(0)\right]|V=0\right) \le 1$$
$$(1-\delta)\left(\Phi(y)-\Phi\left(\frac{2}{3}y\right)\right) \le \Pr\left(Y^n(k+1) \in \left[\theta^n_y, \theta^n_{\frac{2}{3}y}\right]|V=0\right) \le 1$$

for all for all $n > N(\delta)$. Moreover, Lemma A.10 implies that

$$(1-\delta)\frac{\phi\left(x+\frac{y}{\eta}\right)}{\phi(0)}\eta\left(\Phi\left(\frac{y}{3}\right)-\frac{1}{2}\right) \le \Pr\left(Y^n(k+1) \in \left[\theta^n_{\frac{1}{3}y}, \theta^n(0)\right]|V=1\right) \le 1$$
$$(1-\delta)\frac{\phi\left(x+\frac{y}{\eta}\right)}{\phi(0)}\eta\left(\Phi(y)-\Phi\left(\frac{2}{3}y\right)\right) \le \Pr\left(Y^n(k+1) \in \left[\theta^n_y, \theta^n_{\frac{2}{3}y}\right]|V=1\right) \le 1$$

for all $n > N(\delta)$. Therefore,

$$(1-\delta)\frac{\phi\left(x+\frac{y}{\eta}\right)}{\phi(0)}\eta\min\left\{\Phi\left(\frac{y}{3}\right)-\frac{1}{2}, \Phi(y)-\Phi\left(\frac{2}{3}y\right)\right\} \le \frac{\Pr\left(V=1|P^n = b^n(\theta^*)\right)}{\Pr\left(V=0|P^n = b^n(\theta^*)\right)}$$
$$\le \frac{1}{(1-\delta)\min\left\{\Phi\left(\frac{y}{3}\right)-\frac{1}{2}, \Phi(y)-\Phi\left(\frac{2}{3}y\right)\right\}}$$

[31] Observe that $N(\delta)$ is independent of θ^* and the set $\left[\theta^n_{\frac{2}{3}y}, \theta^n_{\frac{1}{3}y}\right]$.

for all for all $n > N(\delta)$. Hence picking ϵ such that $\epsilon < \frac{\phi\left(x+\frac{y}{\eta}\right)}{\phi(0)}\eta \min\left\{\Phi\left(\frac{y}{3}\right)-\frac{1}{2}, \Phi\left(y\right)-\Phi\left(\frac{2}{3}y\right)\right\}$, $\epsilon < \min\left\{\Phi\left(\frac{y}{3}\right)-\frac{1}{2}, \Phi\left(y\right)-\Phi\left(\frac{2}{3}y\right)\right\}$ and $\frac{1}{\epsilon} > \frac{\phi(0)}{\phi(y)}\frac{1}{\eta}$ establishes that $\mathbf{H}$ does not aggregate information. $\square$

We now argue that if there is no pooling by pivotal types and if the pivotal types are distinct, then information is aggregated along a sequence $\mathbf{H}$. Denote by $v \in \{0,1\}$ the state where the pivotal type is largest and denote by $v' \in \{0,1\}$ the other state, i.e., $\theta_s^n(v') < \theta_s^n(v)$. Our assumption that the pivotal types are distinct implies that $\sqrt{n}F_s^n\left([\theta_s^n(v'), \theta_s^n(v)]\,|0\right) \to \infty$. For any $\epsilon \in (0, 1/2)$ define

$$\begin{aligned} \bar{\theta}_\epsilon^n &:= \min\left\{\theta : \Pr(Y_s^n(k+1) \leq \theta | V = v) = \epsilon\right\}, \\ \underline{\theta}_\epsilon^n &:= \max\left\{\theta : \Pr(Y_s^n(k+1) \geq \theta | V = v') = \epsilon\right\}, \\ b_\epsilon^n &:= \frac{b^n\left(\underline{\theta}_\epsilon^n\right) + b^n\left(\bar{\theta}_\epsilon^n\right)}{2} \end{aligned} \tag{A.6}$$

These definitions imply that $\theta_s^n(v') < \underline{\theta}_\epsilon^n < \bar{\theta}_\epsilon^n < \theta_s^n(v)$ for sufficiently large n. This is because $\lim \sqrt{n}F_s^n\left(\left[\bar{\theta}_\epsilon^n, \theta_s^n(v)\right]|V=v\right) \in (0,\infty)$ and $\lim \sqrt{n}F_s^n\left([\theta_s^n(v'), \underline{\theta}_\epsilon^n]\,|V=v'\right) \in (0,\infty)$ by the law of large numbers (or a simple application of Chebyshev's inequality) and $\sqrt{n}F_s^n\left([\theta_s^n(v'), \theta_s^n(v)]\,|0\right) \to \infty$.

We prove the result through the three claims given below. We first argue that $\Pr(Y_s^n(k+1) \leq \underline{\theta}_\epsilon^n|v) \to 0$ and $\Pr(Y_s^n(k+1) \geq \bar{\theta}_\epsilon^n|v') \to 0$ (Claim A.3). We then show that the types $\underline{\theta}_\epsilon^n$ and $\bar{\theta}_\epsilon^n$ submit distinct bids and therefore $b^n\left(\underline{\theta}_\epsilon^n\right) < b^n\left(\bar{\theta}_\epsilon^n\right)$ (Claim A.4). We complete the proof by showing that the bid distribution is state v lies above b_ϵ^n and the bid distribution in state v' lies below b_ϵ^n with probability converging to one, i.e., b_ϵ^n separates the two bid distributions (Claim A.5).

Claim A.3. If $\sqrt{n}F_s^n\left([\theta_s^n(v'), \theta_s^n(v)]\,|0\right) \to \infty$, then $\Pr(Y_s^n(k+1) \leq \underline{\theta}_\epsilon^n|v) \to 0$ and $\Pr(Y_s^n(k+1) \geq \bar{\theta}_\epsilon^n|v') \to 0$.

Proof. Note $\sqrt{n}F_s^n\left(\left[\underline{\theta}_\epsilon^n, \bar{\theta}_\epsilon^n\right]|0\right) \to \infty$ because

$$\lim \sqrt{n}F_s^n\left(\left[\theta_s^n(v'), \theta_s^n(v)\right]|0\right) = \lim \sqrt{n}\left(F_s^n\left(\left[\theta_s^n(v'), \underline{\theta}_\epsilon^n\right]|v'\right) + F_s^n\left(\left[\underline{\theta}_\epsilon^n, \bar{\theta}_\epsilon^n\right]|0\right) + F_s^n\left(\left[\bar{\theta}_\epsilon^n, \theta_s^n(v)\right]|0\right)\right)$$

using $\lim \sqrt{n}F_s^n\left([\theta_s^n(v'), \theta_s^n(v)]\,|0\right) = \infty$, $\lim \sqrt{n}F_s^n\left(\left[\bar{\theta}_\epsilon^n, \theta_s^n(v)\right]|0\right) \in (0,\infty)$ and

$$\lim \sqrt{n}F_s^n\left(\left[\theta_s^n(v'), \underline{\theta}_\epsilon^n\right]|0\right) \in (0,\infty).$$

Moreover, $\sqrt{n}F_s^n\left([\underline{\theta}_\epsilon^n, \theta_s^n(v)]\,|v\right) \to \infty$ and $\sqrt{n}F_s^n\left(\left[\theta_s^n(v'), \bar{\theta}_\epsilon^n\right]|v'\right) \to \infty$ follow immediately from $\sqrt{n}F_s^n\left([\underline{\theta}_\epsilon^n, \theta_s^n(v)]\,|v\right) \geq \sqrt{n}F_s^n\left(\left[\underline{\theta}_\epsilon^n, \bar{\theta}_\epsilon^n\right]|v\right)$ and $\sqrt{n}F_s^n\left(\left[\theta_s^n(v'), \bar{\theta}_\epsilon^n\right]|v'\right) \geq \sqrt{n}F_s^n\left(\left[\underline{\theta}_\epsilon^n, \bar{\theta}_\epsilon^n\right]|v'\right)$. Finally, the law of large numbers implies that $\Pr(Y_s^n(k+1) \leq \underline{\theta}_\epsilon^n|v) \to 0$ and $\Pr(Y_s^n(k+1) \geq \bar{\theta}_\epsilon^n|v') \to 0$. $\square$

Claim A.4. If the pivotal types are distinct and there is no pooling by pivotal types, then $b^n\left(\underline{\theta}_\epsilon^n\right) < b^n\left(\bar{\theta}_\epsilon^n\right)$ for all sufficiently large n.

Proof. Monotonicity implies $b^n\left(\underline{\theta}_\epsilon^n\right) \leq b\left(\bar{\theta}_\epsilon^n\right)$. Suppose $b^{n_k}\left(\underline{\theta}_\epsilon^{n_k}\right) = b^n\left(\bar{\theta}_\epsilon^{n_k}\right) = b_p^{n_k}$ for all n_k along a subsequence. Then, $\lim \Pr\left(P^{n_k} = b_p^{n_k} | V = v\right) \geq \epsilon > 0$ for each $v = 0, 1$ by Claim A.3. However, this means that there is pooling by pivotal types contradicting the assumption of the claim. □

Claim A.5. If the pivotal types are distinct and there is no pooling by pivotal types, then **H** aggregates information.

Proof. Fix any $\epsilon \in (0, 1/2)$. Claim A.4 implies $b^n\left(\underline{\theta}_\epsilon^n\right) < b_\epsilon^n < b^n\left(\bar{\theta}_\epsilon^n\right)$ for sufficiently large n. Given this definition, we have $\Pr\left(P^n \leq b_\epsilon^n | V = v\right) \leq \epsilon$ and $\lim \Pr\left(P^n \geq b_\epsilon^n | V = v'\right) \leq \epsilon$. Moreover,

$$\int_{p<b_\epsilon^n} \frac{\Pr\left(P^n = p | V = v\right)}{\Pr\left(P^n = p | V = v'\right)} \Pr\left(P^n = p | V = v'\right) dp = \int_{p<b_\epsilon^n} \Pr\left(P^n = p | V = v\right) dp \leq \epsilon$$

Therefore,

$$\Pr\left(P^n \in \left\{p < b_\epsilon^n : \frac{\Pr\left(V = v | P^n = p\right)}{\Pr\left(V = v' | P^n = p\right)} > \sqrt{\epsilon}\right\} | V = v'\right) \leq \sqrt{\epsilon}.$$

Hence,

$$\lim \Pr\left(P^n \in \left\{\frac{\Pr\left(V = v | P^n = p\right)}{\Pr\left(V = v' | P^n = p\right)} > \sqrt{\epsilon}\right\} | V = v'\right) \leq \sqrt{\epsilon} + \lim \Pr\left(P^n \geq b_\epsilon^n | V = v'\right) < 2\sqrt{\epsilon}.$$

Finally, for any $\epsilon' > \sqrt{\epsilon}$ we find

$$\lim \Pr\left(P^n \in \left\{\frac{\Pr\left(V = v | P^n = p\right)}{\Pr\left(V = v' | P^n = p\right)} > \epsilon'\right\} | V = v'\right) < 2\sqrt{\epsilon}$$

Because, ϵ is arbitrary, we conclude that

$$\lim \Pr\left(P^n \in \left\{\frac{\Pr\left(V = v | P^n = p\right)}{\Pr\left(V = v' | P^n = p\right)} > \epsilon'\right\} | V = v'\right) = 0$$

and a symmetric argument establishes the result for $V = v$. □

□

Proof of Lemma 2.3. We prove the result through two claims. In the first claim we show that if information is aggregated and the expected price is positive, then the pivotal types must be ordered. In the second claim we show that if the pivotal types are ordered, then price must converge to value.

Claim A.6. If **H** aggregates information and $\lim \mathbb{E}[P^n] > 0$, then $\sqrt{n}\left(F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\theta_s^n(0)|1\right)\right) \to \infty$.

Proof. If **H** aggregates information, then $\sqrt{n}\left|F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\theta_s^n(0)|1\right)\right| \to \infty$ and there is no pooling by pivotal types by Lemma 2.2. Pick a subsequence (abusing notation, we omit the relabeling of this subsequence) and assume, contrary to the claim that $\sqrt{n}\left(F_s^n\left(\theta_s^n(0)|1\right) - F_s^n\left(\theta_s^n(1)|1\right)\right) \to \infty$ along this subsequence. Moreover, suppose that $\lim \mathbb{E}[P^n | V = 0]$ and $\lim \mathbb{E}[P^n | V = 1]$ exist along this subsequence.

Recall the definition of b_ϵ^n given by Equation A.6. The facts that $\mathbf{H}$ aggregates information and $\sqrt{n}\left(F_s^n\left(\theta_s^n(0)|1\right) - F_s^n\left(\theta_s^n(1)|1\right)\right) \to \infty$ together imply that $\lim \mathbb{E}[P^n|V=0] \geq \lim \mathbb{E}[P^n] \geq \lim \mathbb{E}[P^n|V=1]$ and in particular $\lim \mathbb{E}[P^n|V=0] \geq \lim \mathbb{E}[P^n] > 0$. This is because $\mathbb{E}[P^n|V=0] \geq (1-\epsilon)b_\epsilon^n$ and $\mathbb{E}[P^n|V=1\} \leq (1-\epsilon)b_\epsilon^n+\epsilon$ together imply that $\mathbb{E}[P^n|V=0]+\epsilon \geq \mathbb{E}[P^n|V=1]$ for each ϵ. Consider any type that submits a bid equal to b_ϵ^n. We have $\Pr\left(P^n < b_\epsilon^n|V=1\right) \geq 1-\epsilon$ and $\Pr\left(P^n > b_\epsilon^n|V=0\right) \geq 1-\epsilon$ by definition. Therefore, the utility from bidding b_ϵ^n

$$u(b_\epsilon^n|\theta) \geq \Pr(V=1|\theta)\,(1-\epsilon)\,(1-\mathbb{E}[P^n|V=1]) - \Pr(V=0|\theta)\epsilon$$

for any type θ. As ϵ is arbitrary, we conclude that $\lim u(b^n(\theta)|\theta) \geq \Pr(V=1|\theta)\,(1-\lim \mathbb{E}[P^n|V=1])$ for each θ.

For a given $\epsilon \in (0,\kappa_s)$, pick any type $\theta > \theta_s(0) \geq \theta_s(1)$ such that $\bar{F}_s^n(\theta|0) < \epsilon$. Note that $\lim \Pr\left(P^n \leq b^n(\theta)|V=v\right) = 1$ for $v=0,1$. This type wins with probability at least $\kappa_s - \epsilon$ in state $V=0$. This is because if the type θ bids in a pool with $\theta_s(0)$, then the probability of winning is at least $\kappa_s-\epsilon$ in state $V=0$ by Lemma A.3. Otherwise, this type wins with probability one in both states. Therefore,

$$\lim u(b^n(\theta)|\theta) \leq \Pr(V=1|\theta)\,(1-\lim \mathbb{E}[P^n|V=1]) - (\kappa_s-\epsilon)\Pr(V=0|\theta)\lim \mathbb{E}[P^n|V=0] < \\ \Pr(V=1|\theta)\,(1-\lim \mathbb{E}[P^n|V=1])$$

leading to a contradiction. $\square$

Claim A.7. Suppose $\mathbf{H}$ aggregates information. If $\lim \mathbb{E}[P^n] > 0$ or if $\sqrt{n}\left(F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\theta_s^n(0)|1\right)\right) \to \infty$, then $\lim \mathbb{E}[P^n|V=0]=0$ and $\lim \mathbb{E}[P^n|V=1]=1$.

Proof. Information aggregation and $\lim \mathbb{E}[P^n] > 0$ together imply that $\sqrt{n}\left(F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\theta_s^n(0)|1\right)\right) \to \infty$ by the previous claim. Assume to the contrary that $\lim \mathbb{E}[P^n|V=0] > 0$ along a convergent subsequence. Recall $\bar{\theta}_\epsilon^n$ is the type such that $\Pr\left(Y_s^n(k+1) \leq \bar{\theta}_\epsilon^n|V=1\right) = \epsilon$ (Equation A.6) and note that $\lim \Pr\left(Y_s^n(k+1) \leq \bar{\theta}_\epsilon^n|V=0\right) = 1$.

There are two cases to consider: 1) There is an $\epsilon < \eta \lim \mathbb{E}[P^n|V=0]$ and a subsequence such that $\Pr\left(P^n = b^n\left(\bar{\theta}_\epsilon^n\right)|V=1\right) \to 0$; or alternatively 2) $\liminf \Pr\left(P^n = b^n\left(\bar{\theta}_\epsilon^n\right)|V=1\right) > 0$ for all $\epsilon < \eta \lim \mathbb{E}[P^n|V=0]$.

Case 1: Our assumption that $\Pr\left(P^n = b^n\left(\bar{\theta}_\epsilon^n\right)|V=1\right) \to 0$ implies $\lim \Pr(P^n > b^n\left(\bar{\theta}_\epsilon^n\right)|V=1) = \lim \Pr\left(Y_s^n(k+1) > \bar{\theta}_\epsilon^n|V=1\right) = 1-\epsilon$. Therefore,

$$\lim u(b^n\left(\bar{\theta}_\epsilon^n\right)|\bar{\theta}_\epsilon^n) \leq \lim\left(\Pr(V=1|\bar{\theta}_\epsilon^n)\epsilon - \Pr(V=0|\bar{\theta}_\epsilon^n)\mathbb{E}[P^n|V=0]\right).$$

However, $\lim \mathbb{E}[P^n|V=0] > 0$ implies that $\lim u\left(b^n\left(\bar{\theta}_\epsilon^n\right)|\bar{\theta}_\epsilon^n\right) < 0$ because $\epsilon < \eta \lim \mathbb{E}[P^n|V=0]$ and because $\frac{\Pr(V=0|\bar{\theta}_\epsilon^n)}{\Pr(V=1|\bar{\theta}_\epsilon^n)} = \frac{1}{l\left(\bar{\theta}_\epsilon^n\right)} > \eta$ leading to a contradiction. Therefore, $\lim \mathbb{E}[P^n|V=0]=0$.

Case 2: Our assumption $\liminf \Pr\left(P^n = b^n\left(\bar{\theta}_\epsilon^n\right)|V=1\right) > 0$ implies

$$\lim \Pr\left(Y_s^n(k+1) \in \left\{\theta : b^n(\theta) = b^n\left(\bar{\theta}_\epsilon^n\right)\right\}|V=1\right) > 0.$$

In other words, $\bar{\theta}_\epsilon^n$ bids in a pool and $\lim \sqrt{n}F_s^n\left(\left\{\theta : b^n(\theta) = b^n\left(\bar{\theta}_\epsilon^n\right)\right\}|V=1\right) > 0$. However, such a pool is not possible if $\sqrt{n}\left(F_s^n\left(\theta_s^n(1)|1\right)-F_s^n\left(\theta_s^n(0)|1\right)\right)\to\infty$, $\lim\Pr\left(P^n=b^n\left(\bar{\theta}_\epsilon^n\right)|V=1\right)>0$ and $\lim\Pr\left(P^n=b^n\left(\bar{\theta}_\epsilon^n\right)|V=0\right)=0$ by Lemma A.7.

Information aggregation and $\sqrt{n}\left(F_s^n\left(\theta_s^n(1)|0\right)-F_s^n\left(\theta_s^n(0)|0\right)\right)\to\infty$ together imply that $\lim\Pr\left(P^n\leq b^n\left(\theta_s^n(0)\right)|V=1\right)=0$. Therefore, $\lim u(b^n\left(\theta_s^n(0)\right)|\theta_s^n(0))=0$. However,

$$0=\lim u(b^n\left(\theta_s^n(0)\right)|\theta_s^n(0))\geq\lim u(b=1|\theta_s^n(0))=$$
$$\lim\left(\Pr(V=1|\theta_s^n(0))(1-\mathbb{E}[P^n|V=1])\right)\geq 0$$

which implies $\lim\mathbb{E}[P^n|V=1]=1$. □

□

The following lemma also provides conditions for information aggregation that we frequently use in the following development.

Lemma A.11. *Fix a sequence of bidding equilibria* $\mathbf{H}$. *If* $F_s\left(\theta_s(1)|1\right)>F_s\left(\theta_s(0)|1\right)$ *or if* $F_s\left(1|1\right)\geq F_s\left(1|0\right)$ *and* $F_s\left(1|1\right)>\kappa_s$, *then there is no pooling by pivotal types and price converges to value.*

Proof. For the following argument, note that $F_s\left(1|1\right)\geq F_s\left(1|0\right)$ and $F_s\left(1|1\right)>\kappa_s$ together imply that $F_s\left(\theta_s(1)|1\right)>F_s\left(\theta_s(0)|1\right)$ (see Lemma A.5). Under the lemma's assumptions the pivotal types are distinct and pooling by pivotal types is incompatible with equilibrium by Lemma A.5. However, then Lemma 2.2 implies that information is aggregated and Claim A.7 further implies that price converges to value because $F_s\left(\theta_s(1)|1\right)>F_s\left(\theta_s(0)|1\right)$. □

B. The Market Selection Lemmata.

In this section, we characterize market selection. Throughout the section we use $a_s^H(\theta):=a^H(\theta)$ and $a_r^H(\theta):=1-a^H(\theta)$ to simplify exposition. Moreover, in order facilitate a unified exposition, if option r is an exogenous outside option, then we define $u^H(r,b|V=0):=u^H(r|V=0)$ for any b; on the other hand, if option r is an auction market, then $u^H(r,b|V)$ is defined in the usual way as the payoff from bidding b in auction r in state V.

Lemma B.1 (Single Crossing Lemma). *Suppose that* $a_m^H(\theta')>0$ *where* $m\in\{s,r\}$ *for some type* θ' *in an equilibrium* H *. If*

$$u^H(m,b(\theta')|V=0)<u^H(m',b|V=0),$$

for $m\neq m'\in\{s,r\}$ *and some bid* $b\geq 0$*, then*

$$u(m,b(\theta')|\theta)>u(m',b|\theta)$$

for all $\theta>\theta'$ *such that* $\theta\notin\mathcal{E}(\theta')$*.*

Proof. Fix an equilibrium H. For the remainder of the proof we suppress reference to the equilibrium H.

Note that $u(m,b'|\theta,V=v)=u(m,b'|V=v)$ for any b', θ and v. Writing down the profit for type θ from bidding b in market m, we obtain $\frac{u(m,b|\theta)}{\Pr(V=0|\theta)}=u(m,b|V=0)+u(m,b|V=1)l(\theta)$. Therefore,

$$\begin{aligned}\frac{u(m,b(\theta')|\theta)-u(m',b|\theta)}{\Pr(V=0|\theta)}-\frac{u(m,b(\theta')|\theta')-u(m',b|\theta')}{\Pr(V=0|\theta')} &= (l(\theta)-l(\theta'))(u(m,b(\theta')|V=1)\\ &\quad -u(m',b|V=1))\end{aligned}$$

for any two types θ and θ'.

Our initial assumption that $a_m(\theta')>0$ implies $u(m,b(\theta')|\theta')-u(m',b|\theta')\geq 0$. Moreover, $u(m,b(\theta')|\theta')-u(m',b|\theta')\geq 0$ and $u(m,b(\theta')|V=0)<u(m',b|V=0)$ together imply that $u(m,b(\theta')|V=1)-u(m',b|V=1)>0$. Hence, if $\theta>\theta'$ and $\theta\notin\mathcal{E}(\theta')$, then

$$\begin{aligned}&\frac{u(m,b(\theta')|\theta)-u(m',b|\theta)}{\Pr(V=0|\theta)}-\frac{u(m,b(\theta')|\theta')-u(m',b|\theta')}{\Pr(V=0|\theta')}\\ &\qquad\qquad=(l(\theta)-l(\theta'))(u(m,b(\theta')|V=1)-u(m',b|V=1))>0\end{aligned}$$

because $l(\theta)>l(\theta')$ by MLRP. Thus, we find

$$u(m,b(\theta')|\theta)-u(m',b|\theta)>\frac{\Pr(V=0|\theta)}{\Pr(V=0|\theta')}\left(u(m,b(\theta')|\theta')-u(m',b|\theta')\right)\geq 0.$$

□

Below we define $\hat{\theta}_m$ for $m\in\{s,r\}$ as the smallest type which wins a good with positive probability if $V=0$ at the limit as n grows large, i.e., this type is the smallest "active" type in state $v=0$.

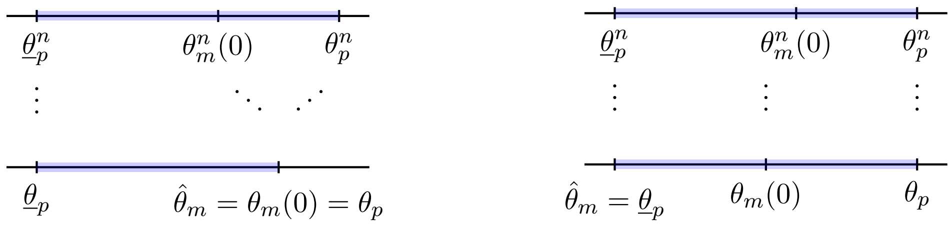
Definition B.1. Fix a sequence of symmetric distributional strategies $\{H^n\}$. If $F_m(1|0)\geq\kappa_m$, let

$$\begin{aligned}\theta^n_m(\epsilon) &:= \inf\left\{\theta : H^n\left([0,1]\times m\times(b^n(\theta),1]\,|0\right)<\kappa_m-\epsilon\right\}\\ \hat{\theta}_m(\epsilon) &:= \limsup\theta^n_m(\epsilon)\\ \hat{\theta}_m &:= \inf_{\epsilon>0}\theta_m(\epsilon).\end{aligned}$$

If $F_m(1|0)<\kappa_m$, let $\hat{\theta}_m=\inf\{\theta:F_m(\theta|0)>0\}$, and $\hat{\theta}_m=1$ if the set is empty.

Suppose that $F_s(1|0)\geq\kappa_s$. The definition above selects type $\hat{\theta}_s=\theta_s(0)$ if the bidding function b^n is strictly increasing at $\theta^n_s(0)$ for sufficiently large n. The definition has more bite if, on the other hand, $\theta^n_s(0)$ submits a pooling bid. If $\theta^n_s(0)$ submits a pooling bid, then there are types $\underline{\theta}^n_p\leq\theta^n_s(0)\leq\theta^n_p$ who submit the same bid as $\theta^n_s(0)$. There are two cases to consider: In the first case $\theta_s(0)=\lim\theta^n_p$. Then the definition selects $\hat{\theta}_s=\theta_s(0)$. In the second case, if $\theta_s(0)<\lim\theta^n_p$, then the definition selects $\hat{\theta}_s=\lim\underline{\theta}^n_p$. Also, see figure B.1 for an illustration.

Lemma B.2. *Assume 5. Suppose that for an equilibrium sequence* $\mathbf{H}$ *we have that* $\lim\mathbb{E}(P^n_s|0)=0$ *and* $\lim\mathbb{E}(P^n_r|0)>0$*, then* $\lim a^n_r(\theta)=1$ *for any* $\theta>\hat{\theta}_r$.



(a) The pivotal type converges to the upper bound of the pooling region.

(b) The pivotal type remains in the interior of the pooling region.

Figure B.1: Implications of Definition B.1. Suppose that $b_m^n(\underline{\theta}_p^n) = b_m^n(\theta_m^n(0)) = b_m^n(\theta_p^n)$ along some sequence. In subfigure (a), $\theta_m(0) = \lim \theta_p^n$, and therefore $\hat{\theta}_m = \theta_m(0)$. In subfigure (b) $\theta_m(0) < \lim \theta_p^n$, and therefore $\hat{\theta}_m = \lim \underline{\theta}_p^n$.

Proof. The fact that $\lim_n \mathbb{E}(P_s^n|V=0) = 0$ implies that $\lim u^n(s, b|V=0) = 0$ for any b. Pick an $\epsilon > 0$ and a sequence of types $\theta^n \in [\hat{\theta}_r^n(\epsilon/2), \hat{\theta}_r^n(\epsilon)]$ such that the limits $\lim \theta^n$, $\lim \hat{\theta}_r^n(\epsilon/2)$, $\lim \hat{\theta}_r^n(\epsilon)$ all exist and $a_r^n(\theta^n) > 0$. The probability that $P_r^n \leq b_r^n(\theta^n)$ converges to one in state 0. Therefore, the probability that θ^n wins an object in state 0 converges to one if this type does not bid in an atom along the sequence. Otherwise, the probability that this type wins is at least $\epsilon/2$ (see Lemma A.3 for this calculation). Hence,

$$\lim u(r, b_r^n(\theta^n)|V=0) \leq -\frac{\epsilon}{2} \lim \mathbb{E}\left[P_r^n|V=0\right] < 0 = \lim u^n(s, b|V=0).$$

Lemma B.1 then implies that $\lim u(r, b_r^n(\theta^n)|\theta) > \lim u(s, b|\theta)$ for any b and any type $\theta > \lim \theta^n$ such that $\theta \notin \mathcal{E}(\lim \theta^n)$ and therefore $a_r(\theta) = 1$. Similarly, if $\theta > \lim \theta^n$ and $\theta \in \mathcal{E}(\lim \theta^n)$, then $a_r^n(\theta) = 1$. This is because we can pick, without loss of generality, a pure and increasing representation of the market selection strategy a_r^n over $\mathcal{E}(\lim \theta^n)$. Since ϵ is arbitrary and $\hat{\theta}_r = \inf_\epsilon \hat{\theta}_r(\epsilon)$ we conclude that $\lim a_r^n(\theta) = 1$ for any $\theta > \hat{\theta}_r$. □

Lemma B.3 (Another Single Crossing Lemma)**.** *Suppose r is an exogenous outside option. If $u^H(s, b^H(\theta')|V=0) < u(r|V=0)$ and if $a_s^H(\theta') > 0$ for some θ', then $a_s^H(\theta) = 1$ for all $\theta > \theta'$. Similarly, if $u^H(s, b^H(\theta')|V=0) > u(r|V=0)$ and if $a_r^H(\theta') > 0$ for some θ', then $a_r^H(\theta) = 1$ for all $\theta > \theta'$. If $u(r|V=0) \geq 0$ and $a_s^H(\theta') > 0$, then $a_s^H(\theta) = 1$ for all $\theta > \theta'$.*

Proof. The first and second parts of the lemma follow immediately from Lemma B.1. We show that if $u(r|V=0) \geq 0$ and $a_s^H(\theta') > 0$, then $a_s^H(\theta) = 1$ for all $\theta > \theta'$. Let $\underline{\theta} = \sup\left\{\theta : \bar{F}_s(\theta|0) = \bar{F}_s(0|0)\right\}$. Pick any $\theta' > \underline{\theta}$ with $a_s^H(\theta') > 0$. First, we note that $b^H(\theta') > 0$ since there cannot be pooling at a bid equal to zero. This is because any type pooling at zero would have an incentive to outbid the pooling bid and win with probability one, conditional on the price being equal to the pooling bid. Let Y denote the event that the price is less than or equal to $b^H(\theta')$ and bidder i who bids $b^H(\theta')$ wins an object. Note that $\mathbb{E}[P_s|Y, 0] > 0$ because $F_s(\theta'|0) - F_s(\underline{\theta}|0) > 0$ by the definitions of θ' and $\underline{\theta}$ and because $b^H(\theta) > 0$ for any $\theta \in (\underline{\theta}, \theta']$. This implies that $u^H(s, b^H(\theta')|V=0) < 0 \leq u(r|V=0)$. Therefore, $a_s^H(\theta) = 1$ for all $\theta > \theta'$ by Lemma B.1. □

Lemma B.4. *If $\lim a_r^n(\theta) = 1$ for all $\theta > \theta_{en}$ and $\kappa_s > \bar{\kappa}_{en}$ for an endogenous outside option (or $\lim a_r^n(\theta) = 1$ for all $\theta > \theta_{ex}$ and $\kappa_s > \bar{\kappa}_{ex}$ for an exogenous outside option), then either*

$\theta_s(0) > \theta_s(1)$ *or* $\kappa_s > F_s(1|1)$*. Alternatively, if* $\lim a_r^n(\theta) = 0$ *for all* $\theta < \theta_{en}$ *and* $\kappa_s < \bar{\kappa}_{en}$ *(or* $\lim a_r^n(\theta) = 0$ *for all* $\theta < \theta_{en}$ *and* $\kappa_s < \bar{\kappa}_{en}$ *for an exogenous outside option), then* $\theta_s(0) < \theta_s(1)$*.*

Proof. We argue that $\lim a_r^n(\theta) = 1$ for all $\theta > \theta_{en}$, $\kappa_s \leq F_s(1|1)$, and $\kappa_s > \bar{\kappa}_{en}$, together imply that $\theta_s(0) - \theta_s(1) > 0$. Let L_1 denote the set of measurable functions $\alpha : [0,1] \to [0,1]$ and consider the following optimization problem:

$$W(\kappa_s, \theta_{en}) = \max_{\alpha \in L_1} \frac{\int_0^{\theta_{en}} \alpha(\theta) dF(\theta|1)}{\int_0^{\theta_{en}} \alpha(\theta) dF(\theta|0)}$$
$$s.t. \int_0^{\theta_{en}} \alpha(\theta) dF(\theta|1) = \kappa_s$$

MLRP implies that $W(\kappa_s, \theta_{en}) = \frac{F([\theta', \theta_{en}]|1)}{F([\theta', \theta_{en}]|0)}$ where θ' is the type such that $F([\theta', \theta_{en}]|1) = \kappa_s$.[32] If $\kappa_s > \bar{\kappa}_{en} = F([\theta^*(\theta_{en}), \theta_{en}]|0) = F([\theta^*(\theta_{en}), \theta_{en}]|1)$, then $\theta' < \theta^*(\theta')$, and MLRP implies $F([\theta', \theta^*(\theta_{en})]|0) > F([\theta', \theta^*(\theta_{en})]|1)$. Therefore, $W(\kappa_s, \theta_{en}) < 1$.

Assume $\kappa_s > F_s(1|1)$ and define $\alpha^*(\theta)$ as the function which is equal to zero for all $\theta \leq \theta_s(1)$ and equal to $a_s(\theta)$ for all $\theta > \theta_s(1)$. This function α^* is feasible for the above maximization problem. Therefore, we obtain

$$\frac{\bar{F}_s(\theta_s(1)|1)}{\bar{F}_s(\theta_s(1)|0)} = \frac{\int_{\theta_s(1)}^{\theta_{en}} a_s(\theta) dF(\theta|1)}{\int_{\theta_s(1)}^{\theta_{en}} a_s(\theta) dF(\theta|0)} = \frac{\int_0^{\theta_{en}} \alpha^*(\theta) dF(\theta|1)}{\int_0^{\theta_{en}} \alpha^*(\theta) dF(\theta|0)} \leq W(\kappa_s, \theta_{en}) < 1.$$

Therefore, $\theta_s(0) > \theta_s(1)$.

We now argue that if $\lim a_r^n(\theta) = 0$ for all $\theta < \theta_{en}$ and $\kappa_s < \bar{\kappa}_{en}$, then $\theta_s(0) < \theta_s(1)$. Define θ' as the type such that $F([\theta', \theta_{en}]|1) = \kappa_s$. Consider the following minimization problem

$$W(\kappa_s, \theta') = \min_{\alpha \in L_1} \frac{\int_{\theta'}^1 \alpha(\theta) dF(\theta|1)}{\int_{\theta''}^1 \alpha(\theta) dF(\theta|0)}$$
$$s.t. \int_{\theta'}^1 \alpha(\theta) dF(\theta|1) = \kappa_s.$$

MLRP implies that $W(\kappa_s, \theta') = \frac{F([\theta', \theta_{en}]|1)}{F([\theta', \theta_{en}]|0)}$. Also, if $\kappa_s < \bar{\kappa}_{en} = F([\theta^*(\theta_{en}), \theta_{en}]|0) = F([\theta^*(\theta_{en}), \theta_{en}]|1)$, then $\theta' > \theta^*(\theta_{en})$, and hence $W(\kappa_s, \theta') > 1$ by MLRP. Define $\alpha^*(\theta)$ as the function that is equal to zero for all $\theta \leq \theta_s(1)$ and equal to $a_s(\theta)$ for all $\theta > \theta_s(1)$. This α^* is feasible for the minimization problem. Therefore,

$$\frac{\bar{F}_s(\theta_s(1)|1)}{\bar{F}_s(\theta_s(1)|0)} = \frac{\int_{\theta_s(1)}^1 a_s(\theta) dF(\theta|1)}{\int_{\theta_s(1)}^1 a_s(\theta) dF(\theta|0)} = \frac{\int_{\theta'}^1 \alpha^*(\theta) dF(\theta|1)}{\int_{\theta'}^1 \alpha^*(\theta) dF(\theta|0)} \geq W(\kappa_s, \theta') > 1.$$

Therefore, $\theta_s(1) > \theta_s(0)$. ◻

C. Proof of Theorem 3.1.

Proof. Information is not aggregated if $\kappa_s > \bar{\kappa}_{ex}$: There are two cases to consider 1) $u(r|0) < 0$ and 2) $u(r|0) \geq 0$.

[32] In other words, the function $\alpha^*(\theta)$, which is equal to one if $\theta \geq \theta'$ and equal to zero otherwise is a maximizer of the problem.

Suppose that $u(r|0) < 0$ and recall that $u(r|1) \leq 1$ by assumption. First, we argue that $\lim \mathbb{E}[P^n] > 0$ whenever this limit exists. If $\lim \mathbb{E}[P^n] = 0$, then all types would select auction s for all sufficiently large n because any type can guarantee a payoff arbitrarily close to one in state $V = 1$ and zero in state $V = 0$ by submitting a bid equal to one in the auction. However, if all types bid in auction s, then price converges to value and $\lim \mathbb{E}[P^n] = 1/2$ leading to a contradiction.

On the way to a contradiction suppose that information is aggregated. If information is aggregated and $\lim \mathbb{E}[P^n] > 0$, then price converges to value by Lemma 2.3. Therefore, $u^n(s, b|\theta) \to 0$ for all θ and all b. The fact that $u^n(s, b|\theta) \to 0$ for all θ and all b implies that all types $\theta > \theta_{ex}$ would opt for the outside option. This is because $\mathbb{E}[u(r|V)|\theta] > 0$ for all $\theta > \theta_{ex}$ by MLRP. However, the fact that $\kappa_s > \bar{\kappa}_{ex}$ implies that $\theta_s(0) > \theta_s(1)$ or $F_s(1|1) < \kappa_s$ because of Lemma B.4. If $F_s(1|1) < \kappa_s$, then price does not converge to value because the price is equal to zero with probability one whenever $V = 1$, as there are more goods than the expected number of bidders. However, this leads to a contradiction because information is aggregation and $\lim \mathbb{E}[P^n] > 0$, together imply that price converges to value (Lemma 2.3).. We now show $\theta_s(0) > \theta_s(1)$ is not compatible with information aggregation either. Pick $\epsilon > 0$ such that $\theta_s(0) - \epsilon > \theta_s(1) + \epsilon$. We know that $\lim \Pr(P^n \geq b^n(\theta_s(0) - \epsilon) | V = 0) = 1$ and $\lim \Pr(P^n \leq b^n(\theta_s(1) + \epsilon) | V = 1) = 1$ by Lemma A.2. We also have $\lim b_s^n(\theta_s(1) + \epsilon) = 1 > \lim b_s^n(\theta_s(0) - \epsilon) = 0$ because $P^n \to V$. However, $\lim b_s^n(\theta_s(1) + \epsilon) = 1 > \lim b_s^n(\theta_s(0) - \epsilon) = 0$ contradicts Lemma 2.1 (monotonicity of b).

Suppose that $u(r|0) \geq 0$. If $u(r|0) \geq 0$, then there is a type θ' such that all types $\theta > \theta'$ select market s and all types $\theta < \theta'$ select the outside option by Lemma B.3. On the way to a contradiction, suppose that information is aggregated. If information is aggregated the pivotal types are distinct (i.e., $\sqrt{n}\,|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| \to \infty$ by Lemma 2.2. Moreover, the fact that all $\theta > \theta'$ select market s implies that $\sqrt{n}\,(F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)) \to \infty$ because of MLRP. But, if information is aggregated and $\sqrt{n}\,(F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)) \to \infty$, then price converges to value by Claim A.7 in the proof of Lemma 2.3. However, if price converges to value, then all types would choose the outside option because $u(r|1) > u(r|0) \geq 0$ leading to a contradiction.

Information is aggregated if $\kappa_s < \bar{\kappa}_{ex}$: In any equilibrium, the set of types that choose r is a subset of $\{\theta : \theta \geq \theta_{ex}\}$. This is because $\mathbb{E}[u(r|V)|\theta] < 0$ for any $\theta < \theta_{ex}$. However, if $\kappa_s < \bar{\kappa}_{ex}$, then $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$ by Lemma B.4. If $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$, then information is aggregated by Lemma A.11. □

D. Proof of Theorem 4.1.

In the following lemma we first characterize behavior in market r under the assumption that $c > 0$, then using this lemma we prove Theorem 4.1.

Lemma D.1. *Assume 5 and $c > 0$. Along any equilibrium sequence, we have $F_r(1|0) < F_r(1|1) \leq \kappa_r$. Moreover, the price in market r converges to c almost surely if $V = 0$ and converges to a random variable $P_r(1)$ if $V = 1$. The random variable $P_r(1)$ is equal to c with probability $q > 0$ and is equal to 1 with the remaining probability. Information is not aggregated because the price is equal to c with positive probability in both states.*

Proof. The following three claims will together prove the result.

Claim D.1. $F_r(1|0) < F_r(1|1) \leq \kappa_r$.

Proof. We will argue that $F_r(1|0) < F_r(1|1)$. If $F_r(1|0) < F_r(1|1)$, then we must have $F_r(1|1) \leq \kappa_r$. This is because $F_r(1|0) < F_r(1|1)$ and $F_r(1|1) > \kappa_r$ together imply that $P_r^n \to 1$ if $V = 1$ by Lemma A.11. But this is not possible because all the bidders in market r would then earn negative profits.

We now show $F_r(1|0) < F_r(1|1)$. First, suppose that $F_r(1|0) > F_r(1|1)$. This implies that $F_s(1|0) < F_s(1|1)$. There are two cases: $F_s(1|1) > \kappa_s$ and $F_s(1|1) \leq \kappa_s$. If $F_s(1|1) > \kappa_s$, then $P_s^n \to 0$ if $V = 0$ by Lemma A.11, and if $F_s(1|1) \leq \kappa_s$, then again $P_s^n \to 0$ if $V = 0$ because $F_s(1|0) < \kappa_s$. However, if $P_s^n \to 0$ when $V = 0$, then $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ by Lemma B.2. However, if $F_r(1|0) < \kappa_r$, then $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ implies that $F_r(1|1) > F_r(1|0)$, which contradicts our initial assumption. On the other hand, if $F_r(1|0) \geq \kappa_r$, then $F_r(1|1) > \kappa_r$ However, $F_r(1|1) > \kappa_r$ and $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ together imply that $P_r^n \to 1$ if $v = 1$ by Lemma A.11, which is not possible.

Second, suppose that $F_r(1|0) = F_r(1|1)$. There are two cases to consider: $F_r(1|1) > \kappa_r$ and $F_r(1|1) \leq \kappa_r$. If $F_r(1|1) > \kappa_r$, then $P_r^n \to 1$ if $V = 1$ by Lemma A.11, which is not possible. Alternatively, If $F_r(1|1) \leq \kappa_r$, then $F_s(1|1) > \kappa_s$. However, $F_s(1|0) = F_s(1|1)$ and $F_s(1|1) > \kappa_s$ together imply by Lemma A.11 that $P_s^n \to 0$ if $V = 0$. However, as argued previously, if $P_s^n \to 0$ if $V = 0$ and if $F_r(1|0) \leq \kappa_r$, then almost all types in market r win an object when $V = 0$ at a price which is at least c. Therefore, $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ by Lemma B.2. Thus, we conclude that $F_r(1|1) > F_r(1|0)$ because $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$. However, this contradicts that $F_r(1|0) = F_r(1|1)$, as we initially assumed. $\square$

Claim D.2. Assume $\lim \sqrt{n}\left(F_r^n(1|1) - \kappa_r\right) > -\infty$ –i.e., there are more bidders than there are objects in market r with positive probability in state 1. We have $b_r^n(\theta) \to 1$ for any type θ that bids in market r.

Proof. For any $\epsilon > 0$, pick θ^n such that $\Pr(Y_r^{n-1}(n\kappa_r) \in (0, \theta^n)|V = 1) \leq \epsilon$, and recall that we use the convention that $Y_r^{n-1}(n\kappa_r) = 0$ if there are fewer than $n\kappa_r + 1$ bidders in market r. We argue that $\lim b_r^n(\theta) = 1$ for any $\theta > \lim \theta^n$. Any type θ^n in this sequence can ensure winning an object by submitting a bid equal to one in the auction. Therefore, we obtain the following inequality which has type θ^n's equilibrium payoff on the left hand side and θ^n's payoff from submitting a bid equal to one on the right hand side:

$$
\begin{aligned}
&\lim\left(1 - \mathbb{E}\left[P_r^n|b_r^n(\theta^n)\ wins, V = 1\right]\right)\Pr(b_r^n(\theta^n)\ wins|V = 1)l\left(\theta^n\right) \\
&\qquad - \lim \Pr\left(b_r^n(\theta^n)\ wins|V = 0\right)c \geq \\
&\lim\left(1 - \mathbb{E}\left[P_r^n|b_r^n(\theta^n)\ wins, V = 1\right]\right)\Pr\left(b_r^n(\theta^n)\ wins|V = 1\right)l\left(\theta^n\right) \\
&\qquad - \lim \Pr\left(b_r^n(\theta^n)\ wins|V = 0\right)c \\
&+ \lim\left(1 - \mathbb{E}\left[P_r^n|b_r^n(\theta^n)\ loses, V = 1\right]\right)\Pr\left(b_r^n(\theta^n)\ loses|V = 1\right)l\left(\theta^n\right) \\
&\qquad - \lim \Pr\left(b_r^n(\theta^n)\ loses|V = 0\right).
\end{aligned}
$$

Therefore,

$$\lim(1-\mathbb{E}[P_r^n|b_r^n(\theta^n)\ loses, V=1]) - \lim\frac{\Pr(b_r^n(\theta^n)\ loses|V=0)}{\Pr(b_r^n(\theta^n)\ loses|V=1)} \quad\leq\quad 0.$$

We consider two cases: (1) $\lim\Pr(P_r^n = b_r^n(\theta^n)|V=1) = 0$ and (2) $\lim\Pr(P_r^n = b_r^n(\theta^n)|V=1) > 0$. Before analyzing these two cases, we find a bound for $\Pr(b_r^n(\theta^n)\ loses|V=0)$. If we define $\delta := \frac{\kappa_r}{F_r^n(1|0)} - 1$, then we find that $\Pr(P_r^n \geq b_r^n(\theta^n)|V=0) \leq e^{-\frac{\delta^2 nF_r^n(1|0)}{2+\delta}}$ by applying Chernoff's inequality (see Janson et al. (2011, Theorem 2.1) or Lemma H.1 in the online appendix). Therefore,

$$\begin{aligned}\Pr(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ loses|V=0) &\leq \Pr(b_r^n(\theta^n)\ loses|V=0)\\ &\leq \Pr(P_r^n \geq b_r^n(\theta^n)|V=0) \leq e^{-\frac{\delta^2 nF_r^n(1|0)}{2+\delta}}.\end{aligned}$$

Case 1: Suppose that $\lim\Pr(P_r^n = b_r^n(\theta^n)|V=1) = 0$. Then $\lim\Pr(P_r^n > b_r^n(\theta^n)|V=1) = \lim\Pr(P_r^n \geq b_r^n(\theta^n)|V=1) > 0$, and $\Pr(b_r^n(\theta^n)\ loses|V=1) = \lim\Pr(P_r^n \geq b_r^n(\theta^n)|V=1) > 0$. Therefore, using the inequality above, we conclude $\lim\mathbb{E}[P_r^n|b_r^n(\theta^n)\ loses, V=1] = \lim\mathbb{E}[P_r^n|P_r^n \geq b_r^n(\theta^n), V=1] = 1$. Thus, $\lim b_r^n(\theta) = 1$ for almost all $\theta > \lim\theta^n$.

Case 2: Suppose that $\lim\Pr(P_r^n = b_r^n(\theta^n)|V=1) > 0$. If $\lim\Pr(P_r^n = b_r^n(\theta^n)|V=1) > 0$, then further below we argue that $\Pr(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ loses|V=1) \geq \frac{A}{\sqrt{n}}$ for some constant A, for all sufficiently large n. Therefore,

$$\begin{aligned}&0 \geq (1-b_r^n(\theta^n))\Pr(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ loses|V=1)l(\theta^n) - \Pr(b_r^n(\theta^n)\ loses|V=0)\\ &0 \geq (1-b_r^n(\theta^n)) - \frac{e^{-\frac{\delta^2 nF_r^n(1|0)}{2+\delta}}}{\frac{A}{\sqrt{n}}}\end{aligned}$$

for sufficiently large n. Taking limits, we find that $\lim b_r^n(\theta^n) = 1$, and thus $\lim b_r^n(\theta) = 1$ for all $\theta \geq \lim\theta^n$.

We now show that if $\lim\Pr(P_r^n = b_r^n(\theta^n)|V=1) > 0$, then $\Pr(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ loses|V=1) \geq \frac{A}{\sqrt{n}}$, for some constant A and for all sufficiently large n. Let $\underline{\theta}_p^n = \inf\{\theta : b_r^n(\theta) = b_r^n(\theta^n)\}$ and $\theta_p^n = \sup\{\theta : b_r^n(\theta) = b_r^n(\theta^n)\}$. Let θ_*^n be such that

$$\Pr[Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n]|V=1] = \frac{\Pr(P_r^n = b_r^n(\theta^n)|V=1)}{2} = \frac{\Pr[Y_r^n(n\kappa_r) \in [\underline{\theta}_p^n, \theta_p^n]|V=1]}{2}.$$

Let X be the random variable which is equal to the expected number of bidders with types in the interval $[\underline{\theta}_p^n, \theta_*^n]$ who bid in market r. We claim that

$$\begin{aligned}\Pr(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ loses|V=1) \geq&\\ &\frac{\Pr(P_r^n = b_r^n(\theta^n)|1)}{2}\frac{\mathbb{E}[X|Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n], V=1]}{n}.\end{aligned} \tag{D.1}$$

This inequality is satisfied because $Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n]$ implies that $P_r^n = b_r^n(\theta^n)$; conditional on the event $Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n]$ at least X bidders, who submitted the same bid as θ^n, are not allocated objects, and there are at most n bidders who submit the same bid as θ^n.

Our assumption of $\lim \Pr(P_r^n = b_r^n(\theta^n)|V=1) > 0$ implies that $\lim \sqrt{n} F_r^n \left[[\underline{\theta}_p^n, \theta_p^n]|V=1\right] > 0$ and thus $\lim \sqrt{n} F_r^n[[\underline{\theta}_p^n, \theta_*^n)|V=1] > 0$. Note that

$$\mathbb{E}\left[X|Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n], V=1\right] \geq n(1-\kappa_r)A^n,$$

where $A^n = \frac{F_r^n[[\underline{\theta}_p^n, \theta_*^n)|1]}{1-F_r^n[[\theta_*^n,1]|1]}$. Let $A = \frac{1}{2}\lim \frac{\Pr(P_r^n = b_r^n(\theta^n)|V=1)}{2}\sqrt{n}A^n$. Note that $A > 0$ because $\lim \sqrt{n} F_r^n[[\underline{\theta}_p^n, \theta_*^n)|1] > 0$. Thus,

$$\lim \frac{\mathbb{E}\left[X|Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n], V=1\right]}{\sqrt{n}} \geq \lim\,(1-\kappa_r)\,\sqrt{n}A^n > 0.$$

The inequality above and $\lim \Pr(P_r^n = b_r^n(\theta^n)|V=1) > 0$ together imply that

$$\mathbb{E}\left[X|Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n], V=1\right] \frac{\Pr\left(P_r^n = b_r^n(\theta^n)|V=1\right)}{2} \geq A\sqrt{n} \tag{D.2}$$

for all sufficiently large n. Combining inequality (D.1) with inequality (D.2) establishes

$$\Pr\left(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ \mathit{loses}|V=1\right) \geq A/\sqrt{n}$$

for all sufficiently large n. □

Claim D.3. The price in market r converges to c almost surely if $V=0$ and converges to a random variable $P_r(1)$ if $V=1$. The random variable $P_r(1)$ is equal to c with probability $q>0$ and is equal to 1 with the remaining probability.

Proof. The fact that the price converges to c almost surely if $V=0$ follows from the law of large numbers and the fact that $F_r(1|0) < \kappa_r$. Also, note that $\lim \sqrt{n}\,(F_r^n(1|1) - \kappa_r) < \infty$. This is because if $\lim \sqrt{n}\,(F_r^n(1|1) - \kappa_r) = \infty$, then the price clears at the bid of some type with probability one in state 1. However, the previous claim showed that $b_r^n(\theta) \to 1$ for all θ. But then $P_r(1) \to 1$, which implies that all bidders make a loss.

The fact that $\lim \sqrt{n}\,(F_r^n(1|1) - \kappa_r) < \infty$ implies that $P_r(1)$ is equal to c with probability $q > 0$. With the remainder of the probability, i.e., with probability $1-q$, the auction clears at the bid of some type θ and $b_r^n(\theta) \to 1$. Therefore, the auction price is equal to 1 with probability $1-q$. □

□

Proof of Theorem 4.1. Fix an equilibrium sequence $\mathbf{H}$. If $c > 0$, then information is not aggregated in market r by Lemma D.1. We now prove the other assertions in the theorem.

Claim. If $c > 0$ and $\kappa_s > \bar{\kappa}_{en}$, then information is not aggregated in market s.

Proof. Assume, on the way to a contradiction, that information is aggregated in market s.

First suppose that $\lim_n \mathbb{E}\left[P_s^n\right] = 0$. Note that $\lim_n \mathbb{E}\left[P_r^n|V=0\right] = c > 0$ by Lemma D.1. Therefore, if $\lim_n \mathbb{E}\left[P_s^n\right] = 0$, then all types would prefer to submit a bid equal to one in market s for all sufficiently large n. But if all types bid in auction s, then $F_s\left(\theta_s(1)|1\right) > F_s\left(\theta_s(0)|1\right)$ and Lemma A.11 implies that $\lim_n \mathbb{E}\left[P_s^n|V=0\right] = 0$, $\lim_n \mathbb{E}\left[P_s^n|V=1\right] = 1$ and therefore

$\lim_n \mathbb{E}[P^n_s] = 1/2$ which contradicts that $\lim_n \mathbb{E}[P^n_s] = 0$. Hence, if information is aggregated in auction, then price converges to value by Lemma 2.3.

The fact that price converges to value in auction s implies that $\lim u^n(s, b^n_s(\theta)|\theta) = 0$ for all θ. We first argue that $\lim a^n_r(\theta) = 1$ for all $\theta > \theta_{en}$. Recall $\hat{\theta}_r$ is the smallest type that wins and object in state 0 in market r (definition B.1). If $\theta > \hat{\theta}_r$, then $\lim a^n_r(\theta) = 1$ by Lemma B.2 because $\lim_n \mathbb{E}[P^n_r|V = 0] > 0$ and $\lim_n \mathbb{E}[P^n_s|V = 0] = 0$. Also, note that $\hat{\theta}_r \leq \theta_{en}$ because if $\hat{\theta}_r > \theta_{en}$, then $\lim_n \mathbb{E}[P^n_r|V = 1] = c$ because $\theta^F_r(1) \leq \theta_{en}$ by Definition 4.1. However, if $\lim_n \mathbb{E}[P^n_r|V = 1] = c$, then $\lim u^n(r, b^n_r(\theta)|\theta) > 0$ for all $\theta \in (\theta_{en}, \hat{\theta}_r)$, contradicting that $\hat{\theta}_r \leq \theta_{en}$.

If $F_s(1|1) < \kappa_s$, then $P^n_s \to 0$ in state 1 showing that information is not aggregated in market s. Instead suppose that $F_s(1|1) \geq \kappa_s$. Lemma B.4 shows that if $\lim a^n_r(\theta) = 1$ for all $\theta > \theta_{en}$ and if $\kappa_s > \bar{\kappa}_{en}$, then $\theta^n_s(0) - \theta^n_s(1) \geq 0$ for all sufficiently large n. If $F_s(1|1) \geq \kappa_s$, then $\theta^n_s(0) - \theta^n_s(1) \geq 0$ for all sufficiently large n; however, this contradicts our initial assumption that information is aggregated in market s. This is because information aggregation in market s implies that $\theta^n_s(1) - \theta^n_s(0) > 0$ for all sufficiently large n. □

Claim. If $c > 0$ and $\kappa_s < \bar{\kappa}_{en}$, then information is aggregated in market s.

Proof. We prove this by looking at two cases. First, assume that $\theta_{en} = \inf\{\theta : \Pr(V = 1|\theta) > c\}$. The fact that $\kappa_s < \bar{\kappa}_{en}$ implies that $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$, even if all $\theta \geq \theta_{en}$ choose market r by Lemma B.4. If $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$, then Lemma A.11 implies that information is aggregated.

Second, assume that $\theta_{en} = \theta^F_r(1)$. Claim F.3 implies that $F_r(1|1) \leq \kappa_r$. However, if $F_r(1|1) \leq \kappa_r$, then Lemma B.4 implies that $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$. If $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$, then Lemma A.11 implies that information is aggregated. □

Claim. Information is aggregated in both markets if $c = 0$.

Proof. For any equilibrium sequence pick a subsequence $\mathbf{H}$ such that the sequence of numbers $\mathbb{E}[P^n_m|V = v]$ have a limit for each $m \in \{s, r\}$ and each v. We will show that for any such convergent sequence, we have $\mathbb{E}[P^n_m|V = v] \to v$ for each m and v which, in turn, implies that $P^n_m \xrightarrow{p} V$ for $m \in \{s, r\}$.

Step 1. The equality $\lim \mathbb{E}[P^n_m|V = 0] = 0$ and the inequality $\lim \mathbb{E}[P^n_{m'}|V = 0] > 0$ cannot be jointly satisfied.

Suppose, without loss of generality, $\lim \mathbb{E}[P^n_s|V = 0] > 0$. The fact that $\lim \mathbb{E}[P^n_s|V = 0] > 0$ implies that $F_s(1|0) \geq \kappa_s$ because otherwise $P^n_s \xrightarrow{p} 0$ by the law of large numbers. The facts that $\lim \mathbb{E}[P^n_r|V = 0] = 0$ and that $\lim \mathbb{E}[P^n_s|V = 0] > 0$ together imply that $\lim a^n_s(\theta) = 1$ for all $\theta > \hat{\theta}_s$ by Lemma B.2. The fact that $\lim a^n_s(\theta) = 1$ for all $\theta > \hat{\theta}_s$ implies that $\bar{F}_s(\hat{\theta}_s|1) = \bar{F}(\hat{\theta}_s|1) > \bar{F}(\hat{\theta}_s|0) = \bar{F}_s(\hat{\theta}_s|0)$. However, $\hat{\theta}_s \leq \theta_s(0)$, and hence $F_s[\theta_s(0)|1] < F_s[\theta_s(1)|1]$. Then we conclude, by Lemma A.11, that $P^n_s \xrightarrow{p} 0$ if $V = 0$, which leads to a contradiction

Step 2. $\lim \mathbb{E}[P^n_m|V = 0] > 0$ for all $m \in \{s, r\}$ is not possible.

$\lim \mathbb{E}[P^n_m|V = 0] > 0$ implies $F_m(1|0) \geq \kappa_m$. Moreover, there is at least one market where $F_m[\theta_m(0)|1] < F_m[\theta_m(1)|1]$. For this market, however, Lemma A.11 implies that $P^n_m \xrightarrow{p} 0$ in state $V = 0$, contradicting the assumption that $\lim \mathbb{E}[P^n_m|V = 0] > 0$.

Step 3. The above two steps have established that $\lim \mathbb{E}[P_m^n|0] = 0$ for all $m \in \{s, r\}$ in any equilibrium. We now show that $\lim \mathbb{E}[P_m^n|V = 0] = 0$ for $m \in \{s, r\}$ implies that $P_m^n \xrightarrow{p} V$ for $m \in \{s, r\}$.

The fact that $\lim \mathbb{E}[P_m^n|0] = 0$ implies that $\lim u^n(m, b_m^n(\theta)|V = 0) = 0$ for all θ. Moreover, $\lim u^n(m, b_m^n(1)|V = 1) = 1 - \lim \mathbb{E}[P_m^n|V = 1]$ for each $m \in \{s, r\}$. Each type θ can mimic type 1's bidding strategy and obtain a payoff identical to type 1 if $V = 1$. Such a type would obtain a payoff equal to zero in the low state at the limit regardless of the strategy that she uses. Therefore,

$$\lim u^n(m, b_m^n(\theta)|V = 1) \geq \lim u^n(m, b_m^n(1)|V = 1)$$

for each θ. However, a symmetric argument from the perspective of type 1 implies that

$$\lim u^n(m, b_m^n(\theta)|V = 1) \leq \lim u^n(m, b_m^n(1)|V = 1)$$

for each θ. These two inequalities show $\lim u^n(m, b_m^n(\theta)|V = 1) = \lim u^n(m, b_m^n(1)|V = 1)$ for each θ. Therefore, if there is a positive mass of bidders in any two markets $m' \neq m$, then we must have that $\lim u^n(m, b_m^n(\theta)|V = 1) = 1 - \lim \mathbb{E}[P_m^n|V = 1] = 1 - \lim \mathbb{E}[P_{m'}^n|V = 1]$ for all θ.

In any equilibrium, total expected utility plus the total revenue must be at most equal to the total available surplus, that is,

$$\lim \int_{[0,1]} u^n(\theta) dF(\theta) + \sum_m \frac{\lim \mathbb{E}[P_m^n|V = 1]}{2} \min\{F_m(1|1), \kappa_m\} \leq \sum_m \min\{F_m(1|1), \kappa_m\} \frac{1}{2}.$$

Noting that $\lim 2u^n(\theta)f(\theta) = f(\theta|1) \lim u(m, b_m(\theta)|V = 1) = f(\theta|1)(1 - \lim \mathbb{E}[P_m^n|V = 1])$ and substituting into the above we obtain

$$\begin{aligned} \int_{[0,1]} (1 - \lim \mathbb{E}[P_m^n|V = 1]) dF(\theta) &\leq \sum_m \min\{F_m(1|1), \kappa_m\}(1 - \lim \mathbb{E}[P_m^n|V = 1]) \\ 1 - \lim \mathbb{E}[P_m^n|V = 1] &\leq (1 - \lim \mathbb{E}[P_m^n|V = 1]) \left(\sum_m \min\{F_m(1|1), \kappa_m\} \right). \end{aligned}$$

However, the fact that $\sum_m \min\{F_m(1|1), \kappa_m\} \leq \kappa_r + \kappa_s < 1$ implies that $1 - \lim \mathbb{E}[P_m^n|V = 1] = 0$. $\square$

$\square$

E. Proofs of Section 5

Proof of Proposition 5.1. We prove the exogenous outside option case through a number of claims. We generalize the argument for the case of an endogenous outside option at the end of the proof.

Claim E.1. In any equilibrium $F_s^n(\mathcal{E}(0)|0) = 1 - g$.

Proof. The fact that $u(r|0) = -c < 0$ and the fact that any type $\theta \in \mathcal{E}(0)$ can guarantee a payoff equal to zero by bidding zero in auction s implies that $a^n(\theta) = 1$ for $\theta \in \mathcal{E}(0)$ and therefore $F_s^n(\mathcal{E}(0)|0) = 1 - g$. $\square$

Claim E.2. If $a_s^n(\theta) = 1$ for all $\theta \in \mathcal{E}(0)$ and $\theta \in \mathcal{E}(1/2)$, then there is a unique bidding equilibrium in auction s where $b^n(\theta) = 0$ for each $\theta \in \mathcal{E}(0)$, $b^n(\theta) = 1$ for each $\theta \in \mathcal{E}(1)$, and $b^n(\theta) = \mathbb{E}\left[V|Y_s^{n-1}(k_s) = \theta, \theta_i = \theta\right]$ for each $\theta \in \mathcal{E}(1/2)$.

Proof. Any type $\theta \in \mathcal{E}(0)$ would never submit a bid that exceeds 0 because they are certain that the value of the object is equal to zero. Similarly, any type $\theta \in \mathcal{E}(1)$ always submits a bid equal to one because they are certain that the value of the object is equal to one.

We first argue that if $F_s^n(\mathcal{E}(1)|1) = x > 0$, then $b^n(\theta) = \mathbb{E}\left[V|Y_s^{n-1}(k_s) = \theta, \theta_i = \theta\right]$ for each $\theta \in \mathcal{E}(1/2)$. This follows from Lemma 2.1 because the bidding distribution has no atoms except at $b = 1$ and $b = 0$. To see that the bid distribution is atomless, for any positive $x < 1 - g$, define an auxiliary type distribution with three distinct signals $\mathcal{E}(0) = [0, 1/3]$, $\mathcal{E}(1/2) = [1/3, 2/3]$, $\mathcal{E}(1) = [2/3, 1]$ that has the following density functions:

$$\begin{aligned} g_x[\mathcal{E}(0)|V = 1] &= 3(1 - g - x) \\ g_x[\mathcal{E}(1/2)|V = 1] &= 3g \\ g_x[\mathcal{E}(1)|V = 1] &= 3x \\ g_x[\mathcal{E}(0)|V = 0] &= 3(1 - g) \\ g_x[\mathcal{E}(1/2)|V = 0] &= 3g \\ g_x[\mathcal{E}(1)|V = 0] &= 0 \end{aligned} \tag{E.1}$$

In this auxiliary type distribution all types $\theta \in \mathcal{E}(1)$, which take the outside option, are assumed to receive a signal $\theta \in \mathcal{E}(0)$. The bid distribution has no atoms except at $b = 1$ and $b = 0$ because any type $\theta \in \mathcal{E}(1/2)$ would not bid in an atom. This follows from the winners'/losers' curse argument that is identical to Pesendorfer and Swinkels (1997). This auxiliary type distribution satisfies the MLRP assumption of Pesendorfer and Swinkels (1997). Therefore, if the uninformed were to bid in an atom, then we would have a contradiction to Lemma 7 in Pesendorfer and Swinkels (1997) that rules out such out atoms.

We now argue if $F_s^n(\mathcal{E}(1)|1) = 0$, then $b^n(\theta) = 1/2 = \mathbb{E}\left[V|Y_s^{n-1}(k_s) = \theta, \theta_i = \theta\right]$ for each $\theta \in \mathcal{E}(1/2)$. Note that any type $\theta \in \mathcal{E}(1/2)$ would always under cut any atom $b > 1/2$ and out bid any atom $b < 1/2$. Therefore, types $\theta \in \mathcal{E}(1/2)$ can bid with an atom only at $b = 1/2$. If the bid distribution is strictly increasing over some interval of types, then Lemma 2.1 implies that $b^n(\theta) = \mathbb{E}\left[V|Y_s^{n-1}(k_s) = \theta, \theta_i = \theta\right] = \mathbb{E}\left[V|\theta_i = \theta\right] = 1/2$. Therefore, we conclude that all types $\theta \in \mathcal{E}(1/2)$ submit a bid equal to 1/2, i.e., $b^n(\theta) = 1/2 = \mathbb{E}\left[V|Y_s^{n-1}(k_s) = \theta, \theta_i = \theta\right]$ for each $\theta \in \mathcal{E}(1/2)$. □

Claim E.3. In any equilibrium $F_s^n(\mathcal{E}(1/2)|1) = g$ and $F_s^n(\mathcal{E}(1)|1) > 0$.

Proof. Assume to the contrary that $F_s^n(\mathcal{E}(1)|1) = 0$, then $b^n(\theta) = 1/2$ for all $\theta \in \mathcal{E}(1/2)$ by the previous claim. However, in this case any type $\theta \in \mathcal{E}(1)$ can get an object from auction s with probability one at an expected price strictly less than 1/2. The expected price is less than 1/2 because all $\theta \in \mathcal{E}(1/2)$ bid 1/2 and because the probability that the number of objects exceeds the number of bidders in the auction is positive. Therefore, if $F_s^n(\mathcal{E}(1)|1) = 0$, then the payoff from participating in auction s strictly dominates the outside option for $\theta \in \mathcal{E}(1)$ and this contradicts $F_s^n(\mathcal{E}(1)|1) = 0$.

Assume, contrary to the above claim, that $F_s^n(\mathcal{E}(1/2)|1) < g$, i.e., some uninformed bidders choose the outside option. Any type $\theta \in \mathcal{E}(1/2)$ obtains a strictly positive payoff in auction s. This is because $u(s, b^n(\theta)|\theta) \geq u(s, b = 0|\theta)$ for any $\theta \in \mathcal{E}(1/2)$. Moreover, $u(s, b = 0|\theta) > 0$ because the number of bidders in auction s is less than the number of objects with positive probability and therefore a type submitting a bid equal to zero obtains an object with positive probability at a price equal to zero in state $V = 1$. However, $\mathbb{E}[u(r|V)|\theta] = 1 - 2c \leq 0$ for any $\theta \in \mathcal{E}(1/2)$ contradicting that any type $\theta \in \mathcal{E}(1/2)$ would choose r instead of market s. $\square$

Claim E.4. The profit for any $\theta \in \mathcal{E}(1)$ of bidding in market s, i.e., $u(s, b = 1|\theta)$, is decreasing in $x = F_s^n(\mathcal{E}(1)|V = 1)$. Therefore, the mass of types $\theta \in \mathcal{E}(1)$, which choose market s, is uniquely determined in equilibrium.

Proof. Take the auxiliary distribution defined in Equation (E.1). Let

$$\gamma_x(\theta) = \left(\frac{G_x(\theta|V=0)}{G_x(\theta|V=1)}\right)^{n-1-k} \left(\frac{1-G_x(\theta|V=0)}{1-G_x(\theta|V=1)}\right)^{k-1}$$

Any type $\theta's$ bid $b^n(\theta) = \frac{1}{1+\gamma_x(\theta)}$. The expected payment of a type $\theta \in \mathcal{E}(1)$ in state $V = 1$, who submits a bid equal to $b = 1 + \epsilon$ in market s, is given by

$$\int_{1/3}^{1} \frac{1}{1+\gamma_x(\theta)} C g_x(\theta|V=1) G_x(\theta|V=1)^{n-1-k} \left(1 - G_x(\theta|V=1)\right)^{k-1} d\theta \tag{E.2}$$

where $C = \frac{(n-1)!}{(n-k-1)!(k-1)!}$. Such a type wins an object with probability one. We will argue that the expected payment of this type is increasing in x. In order to do so, we first show that

$$\mathbb{E}\left[\frac{1}{1+\gamma_x}|V=1\right] = \int_0^1 \frac{C}{1+\gamma_x(\theta)} g_x(\theta|V=1) G_x(\theta|V=1)^{n-1-k} \left(1 - G_x(\theta|V=1)\right)^{k-1} d\theta \tag{E.3}$$

is increasing in x. Further below we show that the cumulative distribution function $\Pr(\gamma_x \leq a|V = 1)$ is a mean preserving spread of $\Pr(\gamma_{x'} \leq a|V = 1)$ for $1 - g \geq x > x'$. Therefore, $\mathbb{E}[\frac{1}{1+\gamma_x}|V = 1] > \mathbb{E}[\frac{1}{1+\gamma_{x'}}|V = 1]$ because $\frac{1}{1+\gamma_x}$ is a convex function. We now argue $\mathbb{E}\left[\frac{1}{1+\gamma_x}|V = 1\right]$ increasing in x implies that the expression given by Equation (E.2) is increasing in x. Note that $G_{x'}(\theta|V = 0) = G_x(\theta|V = 0)$ for all θ by the definition of the auxiliary distributions given by Equation (E.1). Moreover, for all $\theta \leq 1/3$,

$$g_x(\theta|V=1) < g_{x'}(\theta|V=1)$$
$$G_x(\theta|V=1)^{n-1-k} \left(1-G_x(\theta|V=1)\right)^{k-1} < G_{x'}(\theta|V=1)^{n-1-k} \left(1-G_{x'}(\theta|V=1)\right)^{k-1}$$
$$\frac{g_x(\theta|V=0)}{g_x(\theta|V=1)} > \frac{g_{x'}(\theta|V=0)}{g_{x'}(\theta|V=1)}$$

Therefore, $\gamma_x(\theta) > \gamma_{x'}(\theta)$ for all $\theta \in [0, 1/3]$. However, $\gamma_x(\theta) > \gamma_{x'}(\theta)$ and

$$g_x(\theta|V=1)G_x(\theta|V=1)^{n-1-k}\left(1 - G_x(\theta|V=1)\right)^{k-1} \\ < g_{x'}(\theta|V=1)G_{x'}(\theta|V=1)^{n-1-k}\left(1 - G_{x'}(\theta|V=1)\right)^{k-1}$$

together imply

$$\int_0^{1/3} \frac{C}{1+\gamma_x(\theta)} g_x(\theta|V=1)G_x(\theta|V=1)^{n-1-k}\left(1 - G_x(\theta|V=1)\right)^{k-1} d\theta \\ < \int_0^{1/3} \frac{C}{1+\gamma_{x'}(\theta)} g_{x'}(\theta|V=1)G_{x'}(\theta|V=1)^{n-1-k}\left(1 - G_{x'}(\theta|V=1)\right)^{k-1} d\theta$$

which, together with $\mathbb{E}[\frac{1}{1+\gamma_x}|V=1] > \mathbb{E}[\frac{1}{1+\gamma_{x'}}|V=1]$, implies that

$$\int_{1/3}^{1} \frac{C}{1+\gamma_x(\theta)} g_x(\theta|V=1)G_x(\theta|V=1)^{n-1-k}\left(1 - G_x(\theta|V=1)\right)^{k-1} d\theta \\ > \int_{1/3}^{1} \frac{C}{1+\gamma_{x'}(\theta)} g_{x'}(\theta|V=1)G_{x'}(\theta|V=1)^{n-1-k}\left(1 - G_{x'}(\theta|V=1)\right)^{k-1} d\theta$$

establishing the required inequality.

To complete the proof, we show that the cumulative distribution function $\Pr(\gamma_x \leq a|V=1)$ is a mean preserving spread of $\Pr(\gamma_{x'} \leq a|V=1)$ for any for $1-g \geq x > x'$. First note that $\mathbb{E}[\gamma_x|V=1] = 1$ for all $x \leq 1-g$. Therefore, $\mathbb{E}[\gamma_x|V=1] = \mathbb{E}[\gamma_{x'}|V=1]$. For a fix $t \in (0,1)$, let $1 - G_x(\theta_x(t)|V=1) = t$ and note $1 - G_x(\theta_x(t)|V=0) = t - x$. Also, note that $\Pr(\gamma_x \leq \gamma_x(\theta_x(t))|V=1) = Bi(k; n-1, t)$. Therefore, there exists a unique t^* such that

$$G_x(\theta_x(t^*)|V=0)^{n-1-k}\left(1 - G_x(\theta_x(t^*)|V=0)\right)^{k-1} = \\ G_{x'}(\theta_{x'}(t^*)|V=0)^{n-1-k}\left(1 - G_{x'}(\theta_{x'}(t^*)|V=0)\right)^{k-1}$$

Moreover, for any $t < t^*$

$$G_x(\theta_x(t)|V=0)^{n-1-k}\left(1 - G_x(\theta_x(t)|V=0)\right)^{k-1} \\ < G_{x'}(\theta_{x'}(t)|V=0)^{n-1-k}\left(1 - G_{x'}(\theta_{x'}(t)|V=0)\right)^{k-1}$$

and for any $t > t^*$

$$G_x(\theta_x(t)|V=0)^{n-1-k}\left(1 - G_x(\theta_x(t)|V=0)\right)^{k-1} > G_{x'}(\theta_{x'}(t)|V=0)^{n-1-k}\left(1 - G_{x'}(\theta_{x'}(t)|V=0)\right)^{k-1}.$$

Therefore, the distributions functions are single crossing for any $1-g > x > x' > 0$ and hence the cumulative distribution function $\Pr(\gamma_x \leq a|V=1)$ is a mean preserving spread of $\Pr(\gamma_{x'} \leq a|V=1)$. □

The claims above complete the proof of Proposition 5.1 for the case of an exogenous outside option. For the case of an endogenous outside option, all the steps are identical. However, in order to prove that $x = F_s^n(\mathcal{E}(1)|V=1)$ is unique, we have to further show that $u(r, b=1|\theta)$, is increasing in $x = F_s^n(\mathcal{E}(1)|V=1)$. To see this, note that only types $\theta \in \mathcal{E}(1)$ bid in market r and these types submit a bid equal to one. Therefore, $u(r, b=1|\theta) = 1 - \Pr(P_r^n = c|V=1)$. However, $P_r^n = c$ if and only if there are at least as many objects as bidders in market r and the probability of this event is decreasing in the mass of types $\theta \in \mathcal{E}(1)$ that bid in market r, i.e., $1 - g - x$, and therefore increasing in x. $\square$

Proof of Proposition 5.2.

Claim E.5. In any equilibrium $\lim \sqrt{n} F_s^n(\mathcal{E}(1)|V=1) < \infty$.

Proof. If $\lim \sqrt{n} F_s^n(\mathcal{E}(1)|V=1) = \infty$, then the pivotal types are distinct and a direct computation using the bid function given by Equation (5.2) shows that $b^n(\theta^n(1)) \to 1$ and $b^n(\theta^n(0)) \to 0$, i.e., price converges to value. However, if price converges to value, then no type $\theta \in \mathcal{E}(1)$ would choose market s for sufficiently large n. However, this contradicts that $F_s^n(\mathcal{E}(1)|1) > 0$ for each n as proven in Proposition 5.1. $\square$

Claim E.6. If $c > 1/2$, then the price P^n converges in distribution to a random variable P. The distribution functions of $\Pr(P \leq p|V=1)$ and $\Pr(P \leq p|V=0)$ are both atomless and strictly increasing on the interval $[0,1]$. Moreover, $\lim \mathbb{E}[P^n|V=1] = c$ and $\lim \mathbb{E}[P_s^n|V=0] = 1-c$. If $c = 1/2$, then the price converges in probability to $1/2$ in both states.

Proof. Suppose $\frac{\sqrt{n}}{\sigma} \bar{F}_s^n(\mathcal{E}(1)|V=1) = x \in (0,\infty)$ where $\sigma = \sqrt{\kappa_s(1-\kappa_s)}$. For any real number y , pick a type $\theta^n \in \mathcal{E}(1/2)$ such that $\frac{\sqrt{n}}{\sigma}\left(3g(\frac{2}{3} - \theta^n) - \kappa_s\right) = y$. Such a type exists for all sufficiently large n. Note that $\bar{F}_s^n(\theta^n|V=0) = \frac{\sigma y}{\sqrt{n}} + \kappa_s$ and $\bar{F}_s^n(\theta^n|V=1) = \frac{\sigma y}{\sqrt{n}} + \kappa_s + \bar{F}_s^n(\mathcal{E}(1)|V=1)$. This type's bid is given by

$$b^n(\theta^n) = \frac{\frac{\Pr\left(Y_s^{n-1}(k_s)=\theta^n|V=1\right)}{\Pr\left(Y_s^{n-1}(k_s)=\theta^n|V=0\right)}}{1 + \frac{\Pr\left(Y_s^{n-1}(k_s)=\theta^n|V=1\right)}{\Pr\left(Y_s^{n-1}(k_s)=\theta^n|V=0\right)}}.$$

Lemma A.9 implies that

$$b^n(\theta^n) \to \frac{\frac{\phi(y-x)}{\phi(y)}}{1 + \frac{\phi(y+x)}{\phi(y)}} = \frac{\frac{e^{-\frac{(y-x)^2}{2}}}{e^{-\frac{y^2}{2}}}}{1 + \frac{e^{-\frac{(y-x)^2}{2}}}{e^{-\frac{y^2}{2}}}} = \frac{e^{yx - \frac{x^2}{2}}}{1 + e^{yx - \frac{x^2}{2}}} \in (0,1).$$

Moreover, the central limit theorem implies that $\lim \Pr(P \leq p = b^n(\theta^n)|V=0) \to \Phi(y)$ and $\lim \Pr(P \leq p = b^n(\theta^n)|V=1) \to \Phi(y-x)$. Solving for y as a function of p using equation $p = \frac{e^{yx-\frac{x^2}{2}}}{1+e^{yx-\frac{x^2}{2}}}$ we find $y = \frac{1}{x}\left(\ln\frac{p}{1-p} + \frac{x^2}{2}\right)$. Therefore, $\lim \Pr(P \leq p|V=0) = \Phi\left(\frac{1}{x}\left(\ln\frac{p}{1-p} + \frac{x^2}{2}\right)\right)$ and $\lim \Pr(P \leq p|V=1) = \Phi\left(\frac{1}{x}\left(\ln\frac{p}{1-p} + \frac{x^2}{2}\right)\right)$. Therefore, the density of the limit price is given by

$$\frac{d}{dp}\lim \Pr(P \leq p|V=0) = \frac{d}{dp}\Phi\left(\frac{1}{x}\left(\ln\frac{p}{1-p} + \frac{x^2}{2}\right)\right) = \frac{\phi\left(\frac{1}{x}\left(\ln\frac{p}{1-p} + \frac{x^2}{2}\right)\right)}{xp(1-p)}$$

and $\lim \Pr(P \leq p|V=1) = \frac{\phi\left(\frac{1}{x}\left(\ln\frac{p}{1-p}-\frac{x^2}{2}\right)\right)}{xp(1-p)}$. This gives the limit price in closed form and shows that the distribution of the limit price is atomless and strictly increasing on $[0,1]$. More specifically, $\ln\frac{P}{1-P}$ has a normal distribution with mean $-\frac{x^2}{2}$ (or $\frac{x^2}{2}$) and standard deviation x in state $V=0$ (in state $V=1$) .

We now show that the constant x is uniquely determined. Pick $\epsilon > 0$ sufficiently small and note that any type $\theta' \in \left[\frac{2}{3}-\epsilon, \frac{2}{3}\right]$ wins an object with probability one and any type $\theta'' \in \left[\frac{1}{3}, \frac{1}{3}+\epsilon\right]$ wins an object with probability zero. Therefore,

$$\lim u^n(s, b^n(\theta')|\theta') = \frac{1-\lim \mathbb{E}\left[P^n|V=1\right]-\lim \mathbb{E}\left[P^n|V=0\right]}{2}$$
$$\lim u^n(s, b^n(\theta'')|\theta'') = 0$$

Since θ' and θ'' have identical information, $\lim u^n(s, b^n(\theta')|\theta') = \lim u^n(s, b^n(\theta'')|\theta'')$ which, in turn, shows

$$1-\lim \mathbb{E}\left[P^n|V=1\right] = \lim \mathbb{E}\left[P^n|V=0\right].$$

In order, for types $\theta \in \mathcal{E}(1)$ to remain indifferent between the two markets, we must have $\lim \mathbb{E}\left[P^n|V=1\right] = c$ which in turn implies that $\lim \mathbb{E}\left[P^n|V=0\right] = 1-c$. We now show that there is a unique value of x such that

$$\mathbb{E}\left[P|V=0;x\right] := \int_0^1 p d\Phi\left(\frac{1}{x}\left(\ln\frac{p}{1-p}+\frac{x^2}{2}\right)\right) = 1-c.$$

To see that this equation has a unique solution set $P=\frac{Z}{1+Z}$. The variable Z has a lognormal distribution on support $[0,\infty)$ with parameters $-\frac{x^2}{2}$ and x. Theorem 5 in Levy (1973), which provides a sufficient condition for ordering lognormal distributions, implies that the distribution of Z is decreasing in x in the second order stochastic dominance order.[33] Moreover, $P=\frac{Z}{1+Z}$ is an increasing, concave function for $Z \geq 0$. Therefore, $\mathbb{E}\left[P|V=0;x\right]$ is decreasing in x and converges to zero as $x \to \infty$. Moreover, $\mathbb{E}\left[P|V=0;x\right]$ is equal to $1/2$ for $x=0$ (which we argue below) showing that $\lim \mathbb{E}\left[P^n|V=0;x\right] = 1-c$ has a unique solution for any $c \geq 1/2$. The unique solution has $x=0$ if $c=1/2$ and $x>0$ if $c>1/2$.

We now complete the proof by showing that if $x=0$, then the price converges in probability to $1/2$ and $\lim \mathbb{E}\left[P^n|V=0\right] = 1/2$. If $x=0$, then for any $y \in (-\infty,\infty)$ and sequence of $\{\theta^n\}$ such that $\frac{\sqrt{n}}{\sigma}\left(3g(\frac{2}{3}-\theta^n)-\kappa_s\right) = y$ we have $b^n\left(\theta^n\right) \to \frac{\phi(y)/\phi(y)}{1+\phi(y)/\phi(y)} = \frac{1}{2}$. Moreover, $\lim \Pr\left(P \leq p = b^n\left(\theta^n\right)|V=1\right) = \lim \Pr\left(P \leq p = b^n\left(\theta^n\right)|V=0\right) \to \Phi(y)$. Therefore, $\lim \Pr\left(P < p = 1/2|V=v\right) = 0$ and $\lim \Pr\left(P > 1/2|V=v\right) = 0$ for $v=0,1$, i.e., the price converges in probability to $1/2$. Convergence of the expectation also follows because $P \in (0,1)$. □

□

Proof of Example 5.1. Pick $\epsilon < \frac{1-2c}{2}$ and recall that $b_p = c+\epsilon$. If all types $\theta \in \mathcal{E}(1/2)$ choose market s and submit the pooling bid, then the probability of winning conditional on $P=b_p$

[33] Suppose $\ln Z_A$ is distributed $N(\mu_A,\sigma_A)$ and $\ln Z_B$ is distributed $N(\mu_B,\sigma_B)$. Theorem 5 in Levy (1973) proves that Z_A second order stochastically dominates Z_B if $\mu_A > \mu_B$, $\sigma_A < \sigma_B$, and $\mu_A + \frac{\sigma_A^2}{2} \geq \mu_B + \frac{\sigma_B^2}{2}$. Substituting x for the standard deviation and -$x^2/2$ for the mean of the normal distribution then establishes our claim.

converges to $\frac{\kappa_s}{g}$ and $\frac{\kappa_s}{g\frac{(1-q)}{q}}$, in states 1 and 0, respectively. At the limit, the payoff of $\theta \in \mathcal{E}(1/2)$ submitting the pooling bid is given by

$$\Pr(V=1|\theta)\,(1-b_p)\lim\Pr(b_p\; wins|V=1) - \Pr(V=0|\theta)b_p\lim\Pr(b_p\; wins|V=1) =$$
$$(1-b_p)\,q\frac{\kappa_s}{g} - (1-q)\,b_p\frac{\kappa_s q}{g(1-q)} = \frac{q\kappa_s}{g}(1-2c-2\epsilon) > 0.$$

At the limit, the payoff of $\theta \in \mathcal{E}(1/2)$ submitting a bid greater than the pooling bid is given by

$$(1-b_p)\,q - (1-q)\,b_p = q - b_p < 0.$$

Therefore, at the limit, each $\theta \in \mathcal{E}(1/2)$ prefers submitting the pooling bid instead of bidding slightly above the pooling bid and winning with probability one whenever the price is equal to the pooling bid. The fact that each $\theta \in \mathcal{E}(1/2)$ strictly prefers the pooling bid to submitting a bid greater than the pooling bid at the limit implies that these types also prefer the pooling for sufficiently large n. Also, if a type $\theta \in \mathcal{E}(1/2)$ submits a bid less than the pooling bid, then they never win a object at the limit. And, since their payoff at pooling is positive, they prefer the pooling bid to undercutting the pooling bid. Types $\theta \in \mathcal{E}(1)$ opt for the outside option because $b_p > c$ and all types $\theta \in \mathcal{E}(0)$ submit a bid equal to zero. □

F. Proof of Propositions 6.1 and 6.2.

Lemma F.1. *Suppose that $a(\theta') > 0$ implies $a(\theta) = 1$ for all $\theta > \theta'$, i.e., entry into the auction has a cutoff structure. Then the bidding function $b(\theta)$ is increasing.*

Proof. If entry has a cutoff structure and the upper tail of the type distribution chooses auction s, then there can be no atoms in the bid distribution. The argument of Lemma 7 in Pesendorfer and Swinkels (1997) applies immediately in this case. To see this note that if there was a pool, then the winning and losing probabilities at this pool would remain the same if all the bidders were to choose the auction. However, this would then contradict the winner's curse argument of Lemma 7 in Pesendorfer and Swinkels (1997). If there are no pools however, then the bidding function is increasing by Lemma 2.1. □

Proof of Proposition 6.1. Note $u(r|0) \geq 0$ implies that in every equilibrium there is a cutoff type $\hat{\theta}^n$ such that all type above this type select market s and all types below select the outside option by Lemma B.3. This proves proving item i of the proposition. Further below we argue that this cutoff type is unique which will establish the uniqueness of the equilibrium. Item ii of the proposition follows from Lemma F.1.

We now prove items iii and iv of the proposition. In particular, we show that $\lim\sqrt{n}(\kappa_s - F_s^n(1|1)) = \lim\sqrt{n}(\kappa_s - \bar{F}_s^n(\hat{\theta}^n|1)) = x \in (-\infty,\infty]$. Therefore $\kappa_s > \lim F_s^n(1|0)$ by MLRP. A direct calculation shows that there is a sufficiently small $\epsilon > 0$, such that if $\theta > \theta_s(1) - \epsilon$, then $\lim b^n(\theta) = 1$. If $\sqrt{n}(\kappa_s - \bar{F}_s^n(\hat{\theta}^n|1)) \to -\infty$, then the price would converge in probability to 1 if $V = 1$. This is because $\lim\Pr\left[Y_s^{n-1}(k) \geq \max\left\{\hat{\theta}^n, \theta_s(1)-\epsilon\right\}|1\right] = 1$ because $\lim\sqrt{n}\left(\kappa_s - \bar{F}_s^n(\hat{\theta}^n|1)\right) \to -\infty$ and $\lim b^n\left(\max\left\{\hat{\theta}^n, \theta_s(1)-\epsilon\right\}\right) = 1$. However, then no type would choose to bid in the auction because the payoff in the auction converges to zero. Therefore, $\lim\sqrt{n}\left(\kappa_s - \bar{F}_s^n(\hat{\theta}^n|1)\right) = x \in (-\infty,\infty]$, and the probability that there are more goods than bidders in market s in state $v = 1$ converges to a positive constant, i.e., the probability that the price is equal to zero if $v = 1$ is positive. Also, the fact that $\lim \hat{\theta}^n \geq \theta_s(1)$ implies that $\lim b^n(\theta) = 1$ for all $\theta > \lim \hat{\theta}^n$.

We now prove item v. If $V = 1$, then the price converges in probability to a binary random variable that is either equal to one or equal to zero. This is because all bids converge to 1, hence there are only two possible values that the price can take at the limit: the price is equal to one if the auction clears at the bid of any bidder and the price is equal to zero if there are fewer bidders than there are objects. Moreover, the price is equal to zero almost surely if $V = 0$. The calculation for the value of q ensures that type $\hat{\theta}$ is indifferent between the two options and verifying the expression is straightforward.

We now complete the proof by showing that the cutoff type and therefore the equilibrium is unique. Let

$$g(\theta) := \frac{\mathbb{E}[V|Y_s^{n-1}(k) \leq \theta, \theta] - \mathbb{E}[u(r|V)|\theta]}{\Pr(V = 0|\theta)} = l(\theta)(\Pr(Y_s^{n-1}(k) \leq \theta|1) - u(r|1)) - u(r|0)$$

Claim F.1. There is a unique type $\underline{\theta}$ such that $\mathbb{E}[V|Y^{n-1}(k) \leq \theta, \theta] < \mathbb{E}[u(r|V)|\theta]$ for all $\theta < \underline{\theta}$, and $\mathbb{E}[V|Y^{n-1}(k) \leq \theta, \theta] > \mathbb{E}[u(r|V)|\theta]$ for all $\theta > \underline{\theta}$.

Proof. In order to prove the claim, we will show that the function $g(\theta)$ can cross zero at most once. Let

$$\underline{\theta} := \inf\{\theta : g(\theta) \geq 0\}$$

and set $\underline{\theta} = 1$ if the set is empty. Note that if $g(\theta') \geq 0$ for some θ', then $g(\theta) > 0$ for all $\theta > \theta'$ because $l(\theta)$ is nondecreasing and $\Pr(Y^{n-1}(k) \leq \theta|1)$ is strictly increasing in θ. Therefore, $g(\theta) > 0$ for all $\theta > \underline{\theta}$. A similar argument implies $g(\theta) < 0$ for all $\theta < \underline{\theta}$ thus proving the claim. □

Claim. We will now argue that all types above $\underline{\theta}$ bid in the auction and all types below $\underline{\theta}$ take the outside option, i.e, $\underline{\theta} = \hat{\theta}^n$.

Proof. Case i. Suppose that $\underline{\theta} < \hat{\theta}^n$, and therefore that $\mathbb{E}\left[V|Y^{n-1}(k) \leq \hat{\theta}^n, \hat{\theta}^n\right] > \mathbb{E}\left[u(r|V)|\hat{\theta}^n\right]$. For any $\theta < \hat{\theta}^n$, we have $a(\theta) = 0$, and hence

$$u\left(s, b = 0|\theta\right) = \mathbb{E}\left[V|Y^{n-1}(k) \leq \hat{\theta}^n, \theta\right] \leq \mathbb{E}\left[u(r|V)|\theta\right]$$

for all $\theta < \hat{\theta}^n$. Then,

$$\mathbb{E}\left[V|Y^{n-1}(k) \leq \theta, \theta\right] = \Pr\left(Y^{n-1}(k) \leq \theta|V = 1\right) \Pr\left(V = 1|\theta\right) < \mathbb{E}\left[u\left(r|V\right)|\theta\right]$$

for all $\theta < \hat{\theta}^n$ because $\Pr\left(Y^{n-1}(k) \leq \theta|V = 1\right)$ is strictly increasing in θ. This implies that $\mathbb{E}\left[V|Y^{n-1}(k) \leq \theta, \theta\right] < \mathbb{E}\left[u(r|V)|\theta\right]$ for all $\theta \in \left(\underline{\theta}, \hat{\theta}^n\right)$. However, this contradicts the definition of $\underline{\theta}$.

Case ii. Suppose that $\underline{\theta} > \hat{\theta}^n$, and therefore that $\mathbb{E}\left[V|Y^{n-1}(k) \leq \hat{\theta}^n, \hat{\theta}^n\right] < \mathbb{E}\left[u(r|V)|\hat{\theta}^n\right]$. If $\theta' > \hat{\theta}^n$, then $a(\theta') = 1$, and thus $u\left(s, b(\theta')|\theta'\right) \geq \mathbb{E}\left[u(r|V)|\theta'\right]$. Moreover, $u\left(s, b(\theta')|\theta\right) \geq$

$\mathbb{E}[u(r|V)|\theta]$ for $\theta \geq \theta'$ because of MLRP (see Lemma B.3). Therefore, $\lim_{\theta' \downarrow \hat{\theta}^n} u(s, b(\theta')|\theta) = \mathbb{E}\left[V|Y^{n-1}(k) \leq \hat{\theta}^n, \theta\right] \geq \mathbb{E}[u(r|V)|\theta]$ for all $\theta > \hat{\theta}^n$. Consequently,

$$\mathbb{E}\left[V|Y^{n-1}(k) \leq \theta, \theta\right] = \Pr\left(Y^{n-1}(k) \leq \theta | V = 1\right) \Pr(V = 1|\theta) > \mathbb{E}[u(r|V)|\theta]$$

for all $\theta > \hat{\theta}^n$ because $\Pr\left(Y^{n-1}(k) \leq \theta|V = 1\right)$ is strictly increasing in θ. This implies that $\mathbb{E}\left[V|Y^{n-1}(k) \leq \theta, \theta\right] > \mathbb{E}[u(r|V)|\theta]$ for all $\theta \in \left(\hat{\theta}^n, \underline{\theta}\right)$. However, this contradicts the definition of $\underline{\theta}$. □

□

Proof of Proposition 6.2. Fix an equilibrium sequence $\mathbf{H}$. We will argue below that pooling by pivotal types cannot be sustained if $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1)| \to \infty$. However, if there is no pooling by pivotal types and if $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1)| \to \infty$, then information is aggregated by Lemma 2.2. Therefore, once we conclude that pooling cannot be sustained, this conclusion and Theorem 3.1's finding that information is not aggregated together imply that $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1)| < \infty$. Once we've established that the pivotal types are arbitrarily close, then we'll show that there are more bidders in expectation that objects in both states. □

Claim F.2. Assume $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|v) - \bar{F}_s^n(\theta_s^n(0)|v)| \to \infty$, $\lim \Pr\left(P_s^n \leq b_p^n|V = 1\right) = 1$ and $\lim \Pr\left(P_s^n < b_p^n|V = 0\right) = 0$. If $u(r|1) = 1 - c \in (0, 1/2)$ and $u(r|0) \leq -c$, then pooling by pivotal types cannot be sustained.

Proof. On the way to a contradiction assume that there is pooling by pivotal types. If $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|v) - \bar{F}_s^n(\theta_s^n(0)|v)| \to \infty$ and there is pooling by pivotal types, then $\lim \Pr(P^n \leq b_p^n|V = 1) = 1$ and $\lim \Pr(P^n < b_p^n|V = 0) = 0$ by Lemma A.6. Let $b_p := \liminf b_p^n$. We will show that

$$\lim \frac{\Pr\left(b_p^n \; wins, P_s^n = b_p^n|1\right)}{\Pr\left(b_p^n \; wins, P_s^n = b_p^n|0\right)} l\left(\underline{\theta}_p^n\right) \leq 1 < \frac{b_p}{1 - b_p},$$

which implies that

$$\lim \left((1 - b_p) \Pr\left(b_p^n \; wins, P_s^n = b_p^n|1\right) l\left(\underline{\theta}_p^n\right) - b_p \Pr\left(b_p^n \; wins, P_s^n = b_p^n|0\right)\right) < 0$$

along any subsequence where these limits exist. However, then any type sufficiently close to $\underline{\theta}_p$ makes a loss at pooling, which contradicts that there is a pooling interval as claimed.

We now argue that $\lim \frac{\Pr\left(b_p^n \; wins, P_s^n = b_p^n|1\right)}{\Pr\left(b_p^n \; wins, P_s^n = b_p^n|0\right)} l\left(\underline{\theta}_p^n\right) \leq 1$. Lemma A.3 implies that

$$\lim \frac{\Pr\left(b_p^n \; wins, P_s^n = b_p^n|1\right)}{\Pr\left(b_p^n \; wins, P_s^n = b_p^n|0\right)} = \left(\frac{\kappa_s - (F_s(1|1) - F_s(\theta_p|1))}{\kappa_s - (F_s(1|0) - F_s(\theta_p|0))}\right) \left(\frac{F_s(\theta_p|0) - F_s(\underline{\theta}_p|0)}{F_s(\theta_p|1) - F_s(\underline{\theta}_p|1)}\right).$$

Below we show $\frac{F_s(1|1) - F_s(\theta_p|1))}{F_s(1|0) - F_s(\theta_p|0)} > 1$ and therefore $\frac{\kappa_s - (F_s(1|1) - F_s(\theta_p|1))}{\kappa_s - (F_s(1|0) - F_s(\theta_p|0))} < 1$. Also, MLRP implies that $\frac{F_s(\theta_p|0) - F_s(\underline{\theta}_p|0)}{F_s(\theta_p|1) - F_s(\underline{\theta}_p|1)} \lim l\left(\underline{\theta}_p^n\right) \leq 1$ proving the claim.

We now argue that $\frac{F_s(1|1)-F_s(\theta_p|1))}{F_s(1|0)-F_s(\theta_p|0)} > 1$ and therefore $\frac{\kappa_s-(F_s(1|1)-F_s(\theta_p|1))}{\kappa_s-(F_s(1|0)-F_s(\theta_p|0))} < 1$. Note that $b_p \geq c > 1/2$ because otherwise bidding just above the pooling bid delivers a better payoff than outside option r for any type. However, if $b_p > 1/2$, then submitting a bid that exceeds b_p delivers a strictly negative payoff for any type with $l(\theta) \leq 1$. This is because any type such that $F_s(\theta|1) > F_s(\theta_p|1)$ that bids in market s wins an object with probability one in both states and pays a price that strictly exceeds $1/2$. However, this delivers a negative payoff for any type with $l(\theta) \leq 1$. However, if $l(\theta) > 1$ for all $\theta > \theta_p$ that bid in market s, then $\frac{F_s(1|1)-F_s(\theta_p|1))}{F_s(1|0)-F_s(\theta_p|0)} > 1$.

Claim F.3. If $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|v) - \bar{F}_s^n(\theta_s^n(0)|v)| \to \infty$ and $|u(r|v)| < \bar{u}$ for $v = 0, 1$, then there cannot be pooling by pivotal types.

Proof. On the way to a contradiction assume that there is pooling by pivotal types. If $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|v) - \bar{F}_s^n(\theta_s^n(0)|v)| \to \infty$ and there is pooling by pivotal types, then $\lim \Pr(P^n \leq b_p^n|V = 1) = 1$ and $\lim \Pr(P^n < b_p^n|V = 0) = 0$ by Lemma A.6.

Step 1: $b_p := \liminf b_p^n \geq \min\left\{\frac{1-u(r|1)-u(r|0)}{2}, 1 - u\left(r|1\right)\right\}$.

Suppose that $b_p < \min\left\{\frac{1-u(r|1)-u(r|0)}{2}, 1 - u\left(r|1\right)\right\}$, i.e., rearranging $1 - b_p > u\left(r|1\right)$ and $(1 - b_p) - b_p > u(r|1) + u(r|0)$. If $\lim \Pr(P_s^n \leq b_p^n|V = 1) = 1$, then b_p must satisfy the following inequality for any type θ that picks the outside option because otherwise type θ would prefer to bid just above the pooling bid and win an object with probability one in state 1 instead of choosing the outside option:

$$(1 - b_p)l(\theta) - b_p \leq u(r|1)l(\theta) + u(r|0).$$

If $1 - b_p > u\left(r|1\right)$ and $(1 - b_p) - b_p > u(r|1) + u(r|0)$, then

$$(1 - b_p)l(\theta) - b_p > u(r|1)l(\theta) + u(r|0)$$

for any $l(\theta) > 1$ and therefore $\lim a^n(\theta) = 0$ for any type such that $l(\theta) > 1$. However, if $\lim a^n(\theta) = 0$ for all types with $l(\theta) > 1$ then $\lim \bar{F}_s(\theta_s^n(1)|v) - \lim \bar{F}_s(\theta_s^n(0)|v) > 0$ along any sequence where these limits exist. But $\lim \bar{F}_s(\theta_s^n(1)|v) - \lim \bar{F}_s(\theta_s^n(0)|v) > 0$ implies that information is aggregated which contradicts Theorem 3.1.

Step 2: Let $x := \kappa_s - \bar{F}\left(\theta_s^F(1)|0\right) \in (0, 1)$. If $|u(r|v)| < \bar{u} = x/2$ for $v = 0, 1$, then $b_p \geq xb_p > -u\left(r|0\right)$.

The previous step showed $b_p \geq \min\left\{\left(1 - u(r|1) - u(r|0)\right)/2, 1 - u\left(r|1\right)\right\}$. We now argue $x \min\left\{\left(1 - u(r|1) - u(r|0)\right)/2, 1 - u\left(r|1\right)\right\} > -u((r|0)$, which proves this step because $b_p \geq b_p x \geq x \min\left\{\left(1 - u(r|1) - u(r|0)\right)/2, 1 - u\left(r|1\right)\right\}$. If $x\left(1 - u(r|1) - u(r|0)\right)/2 \leq -u(r|0)$, then $x \leq -2u(r|0) + x\left(u(r|1) + u(r|0)\right) < (2 - x)\,\bar{u} + x\bar{u}$ which in turn implies that $x/2 < \bar{u}$, leading to a contradiction. On the other hand, if $x\left(1 - u\left(r|1\right)\right) \leq -u(r|0)$, then $x \leq xu(r|1) - u(r|0) \leq x\bar{u} + \bar{u}$, hence $x/\left(1 + x\right) \leq \bar{u} = x/2$. However, $x/2 < x/\left(1 + x\right)$ because $x \in (0, 1)$ leading to a contradiction.

Step 3. If $\lim \Pr(P_s^n \leq b_p^n|V = 1) = 1$ and $\lim \Pr(P^n < b_p^n|V = 0) = 0$, then pooling by pivotal types is not possible.

Let $\theta_p = \liminf \theta_p^n$. Note that any type $\theta > \theta_p$ that bids in market s wins an object with probability converging to one and pays at least b_p if $V = 0$. However, we then find that all $\theta > \theta_p$

would select market s by Lemma B.3 because $b_p > -u(r|0)$ by step 2. We now look at two cases to establish the result:

Case 1. $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|0) - \bar{F}_s^n(\theta_s^n(0)|0)| < \infty$. If $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|v) - \bar{F}_s^n(\theta_s^n(0)|0)| < \infty$, then the type $\hat{\theta}_s$ defined in Definition B.1 is equal to θ_p (intuitively, $\hat{\theta}_s$ is the smallest type that wins an object in state 0 in market s). Moreover, any type $\theta > \theta_p$ selects market s by the argument in the previous paragraph. This, however, leads to a contradiction because if all $\theta > \theta_p$ select market s and if $\lim |\bar{F}_s^n(\theta_p^n|0) - \bar{F}_s^n(\theta_s^n(0)|0)| = 0$, then $\lim \bar{F}_s(\theta_s^n(1)|0) - \lim \bar{F}_s(\theta_s^n(0)|0) > 0$ by MLRP, and hence $\lim \bar{F}_s(\theta_s^n(1)|0) - \lim \bar{F}_s(\theta_p^n)|0) > 0$ and therefore $\lim \bar{F}_s(\theta_s^n(1)|1) - \lim \bar{F}_s(\theta_p^n)|1) > 0$. However, $\lim \bar{F}_s(\theta_s^n(1)|1) - \lim \bar{F}_s(\theta_p^n)|1) > 0$ contradicts that $\lim \Pr(P_s^n \leq b_p^n|V = 1) = 1$.

Case 2. $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|0) - \bar{F}_s^n(\theta_s^n(0)|0)| = \infty$. In this case, we argue below that the probability of winning a good at pooling is at least $x > 0$. Type $\hat{\theta}_s$ is equal to $\underline{\theta}_p$ and all types above $\hat{\theta}_s$ win an object with probability at least x in market s at a price which is at least equal to b_p. Then however, all $\theta > \hat{\theta}_s$ would select market s by Lemma B.3 because $b_p x > -u(r|V = 0)$. This would, however, imply that information is aggregated in the auction, which leads us to a contradiction.

Continuing with Case 2, we now argue that the expected fraction of objects remaining at pooling is at least x in state $V = 0$, and therefore the probability of winning an object by bidding the pooling bid is also at least x. Note that $\lim \Pr(P^n \leq b_p^n|V = 1) = 1$ implies that $\kappa_s - \bar{F}_s(\theta_p|V = 1) \geq 0$ by the LLN. The fact that all types $\theta > \theta_p$ select market s implies that $\kappa_s - \bar{F}_s(\theta_p|V = 1) = \kappa_s - \bar{F}(\theta_p|V = 1) \geq 0$. Therefore, we find $\theta_p \geq \theta_s^F(1)$. MLRP implies that $\kappa_s - \bar{F}(\theta_p|V = 1) > \kappa_s - \bar{F}(\theta_p|V = 0)$. Also, $\theta_p \geq \theta_s^F(1)$ implies that $\kappa_s - \bar{F}(\theta_p|V = 0) \geq \kappa_s - \bar{F}(\theta_s^F(1)|V = 0) = x$, hence the expected fraction of goods left over to pooling is at least x. The expected fraction of bidders who submit the pooling bid is at most 1. Therefore, Lemma A.3 implies that the probability of winning at pooling is at least x . □

Claim F.4. We have $F_s(1|1) > \kappa_s$ and $F_s(1|0) > \kappa_s$.

Proof. We first show that $F_s(1|1) \geq \kappa_s$. Assume, on the way to a contradiction, that $F_s(1|1) < \kappa_s$. This implies that any type can guarantee an object in state $V = 1$ by submitting a bid equal to 0. However, then no type would choose the outside option r leading to a contradiction.

Now we show that $F_s(1|0) \geq \kappa_s$. Assume, on the way to a contradiction, that $F_s(1|0) < \kappa_s$. Then, $\lim \mathbb{E}[P_s^n|V = 0] = 0 > u(r|0)$. However, if $\lim \mathbb{E}[P_s^n|V = 0] = 0 > u(r|0)$, then the set of types that select the outside option has a cutoff structure with all types that exceed a cutoff selecting the outside option by Lemma B.3. This implies that $F_s(1|0) - F_s(1|1) > 0$ which contradicts our initial assumption that $F_s(1|0) < \kappa_s$ because $F_s(1|1) \geq \kappa_s$.

The fact that the pivotal types are arbitrarily close implies $F_s(1|1) > \kappa_s$ if and only if $F_s(1|0) > \kappa_s$ (see Lemma A.8 for the logic). Below we show that $F_s(1|1) = \kappa_s$ and $F_s(1|0) = \kappa_s$ is not compatible with equilibrium. Therefore, we conclude $F_s(1|1) > \kappa_s$ and $F_s(1|0) > \kappa_s$.

Suppose that $F_s(1|1) = F_s(1|0) = \kappa_s$. In this case, any type θ such that $F_s(\theta|1) > F_s(\theta_s(1)|1) = 0$ who bids in the auction wins an object with probability one in both states. To see this note that $F_s(\theta|1) > F_s(\theta_s(1)|1)$ implies that $\lim \Pr[P^n \leq b^n(\theta)|V = 1] = 1$. Moreover, $\lim \Pr[b^n(\theta)\ wins|P^n \leq b_s^n(\theta), V = 1] = 1$. This follows because the winning chances are smallest if this type submits a pooling bid. In this case $F_s(1|1) = \kappa_s$ and Lemma A.3 together imply that $\lim \Pr\left[b^n(\theta)\ wins | P^n \leq b_s^n(\theta), V=1\right] = 1$. Therefore, such a type wins an object with probability converging to one at a price equal to $\lim \mathbb{E}[P^n|V=1]$ in state $V=1$. In turn, the fact that the pivotal types are arbitrarily close implies that such a type also wins an object with probability converging to one at a price equal to $\lim \mathbb{E}[P^n|V=0]$ in state $V=0$. However, if $\lim \mathbb{E}\left[P^n|V=0\right] < -u(r|0)$, then all $\theta' < \theta$ would select market s by Lemma B.3. This is incompatible with $F_s(1|v) = \kappa_s$, $v = 0, 1$. Similarly, if $\lim \mathbb{E}\left[P_s^n|V=0\right] > -u(r|0)$, then all $\theta' > \theta$ would select market s by Lemma B.3. This is again incompatible with $F_s\left(1|v\right) = \kappa_s$, $v = 0, 1$. Therefore, $\lim \mathbb{E}\left[P^n|0\right] = -u(r|0)$. However, $\lim \mathbb{E}\left[P^n|0\right] = -u(r|0)$ implies that $\lim \mathbb{E}\left[1 - P^n|V=1\right] = u(r|1)$ in order to ensure that there are types willing to select the options as assumed. However, if $\lim \mathbb{E}\left[P^n|V=0\right] = -u(r|0)$ and $\lim \mathbb{E}\left[1 - P^n|V=1\right] = u(r|1)$, then any type θ such that $\mathbb{E}\left[u(r|V)|\theta_i = \theta\right] < 0$ has a negative equilibrium payoff in market s also. This is because any such type wins an object for sure in market s at a price equal to $\lim \mathbb{E}[P^n|V=v]$ in state $v = 0, 1$ and $\lim \mathbb{E}[v - P^n|\theta] = \mathbb{E}[u(r|V)|\theta] < 0$. However, this leads to a contradiction because any type can guarantee a payoff equal to zero by bidding zero in market s. Therefore, we conclude that $F_s(1|1) > \kappa_s$ and $F_s(1|0) > \kappa_s$. □

□

G. Proof of Proposition 6.3

Proof of Proposition 6.3 . The assertions concerning market r follow from Lemma D.1.

Claim G.1. If $c < \bar{c}$ or if $c > 1/2$, then $\lim_{n\to\infty} \sqrt{n}\left|F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\theta_s^n(0)|1\right)\right| < \infty$.

Proof. Suppose instead that $\lim_{n\to\infty} \sqrt{n}\left|F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\theta_s^n(0)|1\right)\right| \to \infty$. We will argue below that if $c < \bar{c}$ or if $c > 1/2$, then pooling by pivotal types cannot be sustained. However, if there is no pooling by pivotal types and if $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1)| \to \infty$, then information is aggregated. Therefore, once we conclude that pooling by pivotal types cannot be sustained, this conclusion and Theorem 4.1's finding that information is not aggregated together imply that $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1)| < \infty$.

If $c > 1/2$, then pooling by pivotal types is not possible by Claim F.2 given as part of the proof of Proposition 6.2.

Suppose that $c < \bar{c}$ and there is pooling by pivotal types. If there is pooling by pivotal types and $\lim_{n\to\infty} \sqrt{n}\left|F_s^n\left(\theta_s^n(1)|1\right) - F_s^n\left(\theta_s^n(0)|1\right)\right| \to \infty$, then $\lim \Pr(P_s^n \leq b_p^n|V=1) = 1$ and $\lim \Pr(P_s^n < b_p^n|V=0) = 0$ by A.6. Let $\liminf b_p^n = b_p$. We first argue that

$$\frac{b_p}{1-b_p} \geq \lim \left(\frac{\Pr(b_p^n\ loses, P_s^n = b_p^n|V=1)}{\Pr(b_p^n\ loses, P_s^n = b_p^n|V=0)} l(\theta^n)\right) \tag{G.1}$$

for each θ^n that submits the pooling bid b_p^n along any subsequence where the limits exist. For any type θ who submits the pooling bid, the pooling bid must be at least as profitable as submitting a bid slightly higher than the pooling bid, i.e., a bid $b_p^n + \epsilon$ for any $\epsilon > 0$. Therefore:

$$\begin{aligned}\Pr(b_p^n\ wins|V=1, P_s^n = b_p^n)\Pr(P_s^n = b_p^n|V=1)(1-b_p^n)l(\theta)&\\ - \Pr(b_p^n\ wins|V=0, P_s^n = b_p^n)\Pr(P_s^n = b_p^n|V=0)b_p^n \geq&\\ -b_p^n\Pr(P_s^n = b_p^n|V=0) + (1-b_p^n)\Pr(P_s^n = b_p^n|V=1)l(\theta),&\end{aligned}$$

where we ignore ϵ as it is arbitrarily small. Rearranging we find that

$$\frac{b_p^n}{1-b_p^n} \geq \frac{\Pr(b_p^n \ loses, P_s^n = b_p^n|V=1)}{\Pr(b_p^n \ loses, P_s^n = b_p^n|V=0)} l(\theta).$$

We now argue that $b_p \lim \Pr(b_p^n \ wins, P_s^n = b_p^n|V=0) \leq c$, $(1-b_p)\lim \Pr(b_p^n \ wins, P_s^n = b_p^n|V=1) \leq 1-c$ and therefore

$$\frac{b_p}{1-b_p} \leq \frac{c}{1-c} \lim \frac{\Pr(b_p^n \ wins, P_s^n = b_p^n|V=1)}{\Pr(b_p^n \ wins, P_s^n = b_p^n|V=0)}. \tag{G.2}$$

Suppose that $b_p \lim \Pr(b_p^n \ wins, P_s^n = b_p^n|V=0) > c$, then any $\theta > \underline{\theta}_p$, and therefore any type $\theta > \theta_s(1)$, would select market s by Lemma B.1. If so, then MLRP implies that $F_s(\theta_s(1)|1) < F(\theta_s(1)|0)$, which leads to a contradiction because in this case information would be aggregated by Lemma A.11. Suppose that $(1-b_p)\lim \Pr(b_p^n \ wins, P_s^n = b_p^n|V=1) > 1-c$. If this inequality were true, then any type would prefer the pooling bid to market r because state-by-state profits are higher at the pooling bid, which again leads to a contradiction.

The above argument showed that inequalities G.1 and G.2 must be satisfied if there is pooling by pivotal types. We now argue that inequalities G.1 and G.2 together imply that $\frac{c}{1-c} \geq \frac{\bar{c}}{1-\bar{c}}$. The fact that $\lim \Pr(P_s^n \leq b_p^n|V=1) = 1$ implies that $b^n(\theta_s^n(1)) = b_p^n$ for sufficiently large n, which establishes inequality G.1 for $\theta^n = \theta_s^n(1)$. Therefore, inequalities G.1 and G.2 imply

$$\begin{aligned}\frac{c}{1-c} &\geq \lim \frac{\Pr(b_p^n \ loses, P_s^n = b_p^n|V=1)}{\Pr(b_p^n \ wins, P_s^n = b_p^n|V=1)} \frac{\Pr(b_p^n \ wins, P_s^n = b_p^n|V=0)}{\Pr(b_p^n \ loses, P_s^n = b_p^n|V=0)} l(\theta_s^n(1)) \\ &= \lim \frac{\Pr(b_p^n \ loses, P_s^n = b_p^n|V=1)}{\Pr(b_p^n \ wins, P_s^n = b_p^n|V=1)} \frac{\Pr(b_p^n \ wins, P_s^n = b_p^n|V=0)}{\Pr(b_p^n \ loses, P_s^n = b_p^n|V=0)} l(\theta_s^n(1)).\end{aligned}$$

There are two possibilities: either $F_s(\underline{\theta}_p|1) = 0$, or $F_s(\underline{\theta}_p|1) > 0$.

Case 1: Suppose that $F_s(\underline{\theta}_p|1) = 0$. In this case, Lemma A.3 and $F_s(\underline{\theta}_p|1) = 0$ together imply that

$$\begin{aligned}\lim \frac{\Pr(b_p^n \ loses, P_s^n = b_p^n|V=1)}{\Pr(b_p^n \ wins, P_s^n = b_p^n|V=1)} &= \frac{F_s(1|1) - \kappa_s}{\kappa_s - (F_s(1|1) - F_s(\theta_p|1))} \\ \lim \frac{\Pr(b_p^n \ wins, P_s^n = b_p^n|V=0)}{\Pr(b_p^n \ loses, P_s^n = b_p^n|V=0)} &= \frac{\kappa_s - (F_s(1|0) - F_s(\theta_p|0))}{F_s(1|0) - \kappa_s}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{c}{1-c} &\geq \frac{F_s(1|1) - \kappa_s}{F_s(1|0) - \kappa_s} \frac{\kappa_s - (F_s(1|0) - F_s(\theta_p|0))}{\kappa_s - (F_s(1|1) - F_s(\theta_p|1))} \lim l(\theta_s^n(1)) \\ &\geq \frac{(1-\kappa_r-\kappa_s)^2}{1-\kappa_s} \geq \frac{\bar{c}}{1-\bar{c}},\end{aligned}$$

where the second inequality is satisfied because $F_r(1|1) \leq \kappa_r$, $\frac{\kappa_s - (F_s(1|0) - F_s(\theta_p|0))}{\kappa_s - (F_s(1|1) - F_s(\theta_p|1))} \geq 1$, and $\lim l(\theta_s^n(1)) \geq 1 - \kappa_r - \kappa_s$. To see $\lim l(\theta_s^n(1)) \geq 1 - \kappa_r - \kappa_s$, note that

$$\lim l(\theta_s^n(1)) \geq \frac{F(\theta_s(1)|1)}{F(\theta_s(1)|0)} \geq F(\theta_s(1)|1) \geq F_s(\theta_s(1)|1) \geq F_s(1|1) - \kappa_s \geq 1 - \kappa_r - \kappa_s,$$

We now argue that $\frac{(F_s(1|0)-F_s(\theta_p|0))}{(F_s(1|1)-F_s(\theta_p|1))} \leq 1$, which implies $\frac{\kappa_s-(F_s(1|0)-F_s(\theta_p|0))}{\kappa_s-(F_s(1|1)-F_s(\theta_p|1))} \geq 1$. Inequality G.1, Lemma A.3, and $F_s(\underline{\theta}_p|1) = 0$ together imply that

$$\frac{b_p}{1-b_p} \geq \lim \Pr(b_p^n \; loses, P_s^n = b_p^n|V=1) l(\theta_s^n(1)) \geq \frac{(1-\kappa_r-\kappa_s)^2}{1-\kappa_r} \geq \frac{\bar{c}}{1-\bar{c}}.$$

Therefore, if $\frac{c}{1-c} < \frac{\bar{c}}{1-\bar{c}}$, then $b_p > c$, and any type $\theta > \theta_p$ would select market s by Lemma B.1. This implies that $\frac{(F_s(1|0)-F_s(\theta_p|0))}{(F_s(1|1)-F_s(\theta_p|1))} = \frac{(F(1|0)-F(\theta_p|0))}{(F(1|1)-F(\theta_p|1))} < 1$, as claimed. Thus pooling cannot be sustained if $\frac{c}{1-c} < \frac{\bar{c}}{1-\bar{c}}$ and $F_s(\underline{\theta}_p|1) = 0$.

Case 2: Suppose instead that $F_s\left(\underline{\theta}_p|1\right) > 0$. In this case, Lemma A.3 and $F_s\left(\underline{\theta}_p|1\right) > 0$ together imply that

$$\begin{aligned} \lim \frac{\Pr(b_p^n \; loses, P_s^n = b_p^n|V=1)}{\Pr(b_p^n \; wins, P_s^n = b_p^n|V=1)} &= \frac{F_s(\theta_s(1)|1) - F_s(\underline{\theta}_p|1)}{\kappa_s - (F_s(1|1) - F_s(\theta_p|1))} \\ \lim \frac{\Pr(b_p^n \; wins, P_s^n = b_p^n|V=0)}{\Pr(b_p^n \; loses, P_s^n = b_p^n|V=0)} &\geq \frac{\kappa_s - (F_s(1|0) - F_s(\theta_p|0))}{F_s(1|0) - \kappa_s}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{c}{1-c} &\geq \frac{F_s(\theta_s(1)|1) - F_s(\underline{\theta}_p|1)}{F_s(1|0) - \kappa_s} \frac{\kappa_s - (F_s(1|0) - F_s(\theta_p|0))}{\kappa_s - (F_s(1|1) - F_s(\theta_p|1))} \lim l(\theta_s^n(1)) \\ &\geq \frac{x\,(1-\kappa_r-\kappa_s)}{1-\kappa_s}, \end{aligned}$$

where $x = F_s(\theta_s(1)|1) - F_s(\underline{\theta}_p|1)$. In establishing the final inequality, we used the fact that $\frac{\kappa_s-(F_s(1|0)-F_s(\theta_p|0))}{\kappa_s-(F_s(1|1)-F_s(\theta_p|1))} \geq 1$. We provide an argument for this further below.

Any type such that $F_s\left(\theta^n|1\right) = F_s\left(\underline{\theta}_p^n|1\right) - \epsilon$ for $\epsilon > 0$, who bids in market s, has a payoff equal to zero. Any such type can submit the pooling bid. Therefore,

$$\lim \left(\left(1-b_p^n\right) \Pr(b_p^n \; wins, P_s^n = b_p^n|1) \Pr\left(1|\theta^n\right) - b_p^n \Pr(b_p^n \; wins, P_s^n = b_p^n|0) \Pr\left(0|\theta^n\right)\right) \leq 0$$

Rearranging, we conclude

$$\frac{b_p}{1-b_p} \geq \lim \left(\frac{\Pr(b_p^n \; wins, P_s^n = b_p^n|V=1)}{\Pr(b_p^n \; wins, P_s^n = b_p^n|V=0)} l\left(\theta^n\right)\right).$$

Note that inequality G.2 implies that

$$\frac{b_p}{1-b_p} \leq \frac{c}{1-c} \lim \frac{\Pr(b_p^n \; wins, P_s^n = b_p^n|V=1)}{\Pr(b_p^n \; wins, P_s^n = b_p^n|V=0)}.$$

Combining these two inequalities, observing that $\lim l\left(\theta^n\right) \geq F_s\left(\underline{\theta}_p^n|1\right) - \epsilon \geq 1 - \kappa_s - \kappa_r - x - \epsilon$ and using the fact that ϵ is arbitrary we conclude

$$\frac{c}{1-c} \geq 1 - \kappa_s - \kappa_r - x.$$

Therefore,

$$\frac{c}{1-c} \geq \max\left\{1-\kappa_s-\kappa_r-x, \frac{x\left(1-\kappa_r-\kappa_s\right)}{1-\kappa_s}\right\} \geq \frac{\left(1-\kappa_r-\kappa_s\right)^2}{1-\kappa_s+1-\kappa_s-\kappa_r} \geq \frac{\bar{c}}{1-\bar{c}},$$

where we obtain the lower bound by solving for the value of x that minimizes the expression inside the maximum function.[34]

We now complete the argument by showing that $\frac{\kappa_s-(F_s(1|0)-F_s(\theta_p|0))}{\kappa_s-(F_s(1|1)-F_s(\theta_p|1))} \geq 1$. Any type such that $F_s\left(\theta^n|1\right) = F_s\left(\underline{\theta}_p^n|1\right) - \epsilon$ for $\epsilon > 0$ who bids in market s has a payoff equal to zero. Any such type can bid $b_p + \epsilon$ and win an object with probability that converges to one. Thus, we must have $\lim\left(1-b_p^n\right)l(\theta^n) - b_p^n \leq 0$ for any such type. Therefore,

$$\frac{b_p}{1-b_p} \geq \lim l\left(\theta^n\right) \geq F_s\left(\underline{\theta}_p^n|1\right) \geq 1-\kappa_s-\kappa_r-x$$

because ϵ is arbitrary. Moreover, Inequality G.1, Lemma A.3 and $F_s(\underline{\theta}_p|1) > 0$ together imply that

$$\frac{b_p}{1-b_p} \geq \lim \Pr(b_p^n \; loses, P_s^n = b_p^n|V=1)l(\theta_s^n(1)) \geq \frac{x\left(1-\kappa_r-\kappa_s\right)}{1-\kappa_r}.$$

Therefore,

$$\frac{b_p}{1-b_p} \geq \max\left\{1-\kappa_s-\kappa_r-x, \frac{x\left(1-\kappa_r-\kappa_s\right)}{1-\kappa_r}\right\} \geq \frac{\left(1-\kappa_s-\kappa_r\right)^2}{1-\kappa_r+1-\kappa_s-\kappa_r} \geq \frac{\bar{c}}{1-\bar{c}},$$

where to obtain the lower bound we solve for x by observing that inside the maximum we have two linear functions of x, one increasing and the other decreasing in x. Therefore, if $\frac{c}{1-c} < \frac{\bar{c}}{1-\bar{c}}$, then $b_p > c$, and any type $\theta > \theta_p$ would select market s by Lemma B.1. This implies that $\frac{(F_s(1|0)-F_s(\theta_p|0))}{(F_s(1|1)-F_s(\theta_p|1))} = \frac{(F(1|0)-F(\theta_p|0))}{(F(1|1)-F(\theta_p|1))} < 1$ as claimed. □

Claim G.2. *If $c < \bar{c}$, then $\lim \mathbb{E}[P_r^n|V=1] = \lim \mathbb{E}[P_s^n|V=1]$ and $\lim \mathbb{E}[P_r^n|V=0] = \lim \mathbb{E}[P_s^n|V=0] = c$.*

Proof. Note that $F_r\left(1|0\right) < \kappa_r$ implies that $\lim \mathbb{E}[P_r^n|V=0] = c$. If $c > \lim \mathbb{E}[P_s^n|V=0]$, then $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ by Lemma B.2. This implies that $F_s\left(\theta_s(0)|1\right) > F_s\left(\theta_s(1)|1\right)$ because $\kappa_s > \bar{\kappa}_{en}$ and because $\hat{\theta}_r = \theta_{en}$, which contradicts that the pivotal types are arbitrarily close.

Further below we show any type $\theta > \theta_s(0)$ who submits a bid in market s wins an object with probability one if $V=0$. If $c < \lim \mathbb{E}[P_s^n|V=0]$, then $a_s(\theta) = 1$ for all $\theta > \theta_s(0)$ by Lemma B.2. However, this implies that $F_s\left(\theta_s(1)|1\right) > F_s\left(\theta_s(0)|1\right)$, which again contradicts that the pivotal types are arbitrarily close.

We now show that for any type $\theta' > \theta_s(0)$, $\lim \Pr(b_s^n(\theta') \; wins, P_s^n \leq b_s^n(\theta')|V=0) = 1$. Suppose that $\theta' > \theta_s(0)$ and $\lim \Pr(b_s^n(\theta') \; wins, p_s^n \leq b_s^n(\theta')|V=0) < 1$. The law of large numbers implies that $\lim \Pr(P_s^n \leq b_s^n(\theta_s^n(0))|V=0) = 1$. Moreover, monotonicity of bidding implies that $b_s^n(\theta') \geq b_s^n(\theta_s^n(0))$. Therefore, it must be the case that $b_s^n(\theta') = b_s^n(\theta_s^n(0)) = b_p^n$ for all sufficiently large n. Hence there must be a sequence of pooling regions $(\underline{\theta}_p^n, \theta_p^n)$ such that $\underline{\theta}_p \leq \theta_s(0) = \theta_s(1) < \theta' \leq \theta_p$. Our assumption that $\lim \Pr(b_s^n(\theta') \; wins, p_s^n \leq b_s^n(\theta')|V=0) < 1$

[34] Observe that, inside the maximum, we have two linear functions, one increasing and the other decreasing in x. Therefore, this expression in minimized at the value of x where the two expressions are equal.

implies that $\underline{\theta}_p < \theta_s(0)$ because otherwise all bidders in the pooling region would win a good with probability one by Lemma A.3. If $\underline{\theta}_p < \theta_s(0) = \theta_s(1) < \theta_p$, then $\lim \Pr\left(P_s^n \leq b_p^n | V = 1\right) = 1$ and $\lim \Pr\left(P_s^n < b_p^n | V = 0\right) = 0$. However, in claim F.2 we established that such a sequence of pooling bids does not exist.

We now show that $\lim \mathbb{E}[P_s^n | V = 0] = \lim \mathbb{E}[P_r^n | V = 0] = c$ implies $\lim \mathbb{E}[P_r^n | V = 1] = \lim \mathbb{E}[P_s^n | V = 1]$. Note that any type $\theta > \hat{\theta}_r$ who bids in market r wins an object with probability converging to one in both states. If, however, $\lim \mathbb{E}[P_r^n | V = 1] > \lim \mathbb{E}[P_s^n | V = 1]$, then any such type would prefer to submit a bid equal to one in market s. Similarly, we argued above that any type $\theta' > \theta_s(0)$ wins an object with probability one in both states. However, if $\lim \mathbb{E}[P_r^n | V = 1] < \lim \mathbb{E}[P_s^n | V = 1]$, then any such type would prefer to submit a bid equal to one in market r. □

□

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H. Online Appendix

Below we state a number of statistical results that we frequently use.

Proposition H.1. *(Chernoff's Inequality) Suppose that $X \sim bi(n,p)$, i.e., X is a binomial random variable with probability of success equal to p, then for any $\delta \in (0,1)$*

$$\Pr\left(X \geq (1+\delta)np\right) \leq \exp\left(-\frac{\delta^2 np}{2+\delta}\right),$$
$$\Pr\left(X \leq (1-\delta)np\right) \leq \exp\left(-\frac{\delta^2 np}{2}\right).$$

Proof. See Janson et al. (2011, Theorem 2.1). □

Proof of Lemma 2.1: Bidding is nondecreasing in any bidding equilibrium. We show that any bidding equilibrium H can be represented by a nondecreasing bidding function b.

Fix a bidding equilibrium H for market m. Suppose there is a positive probability under H of a bid strictly above $V = 1$. Then there is a positive probability that $k+1$ bids, and thus the price is strictly larger than 1. But any bid that wins with positive probability at a price above 1 is strictly worse than a bid of 1 and we can conclude that bids are always less than 1.

Suppose there is a positive probability under H of a bid strictly below $V = 0$. Then a bid of 0 wins with strictly greater probability than any bid strictly below 0 if $V = 1$ and makes strictly positive profit if $V = 1$. We can then conclude that bids are always greater than 0.

Take any $b' < b < 1$, let $Y(b,b')$ denote the event where a player wins an object with bid b and does not win an object with bid b', and suppose $\Pr(Y(b,b')) > 0$. Take any $\theta > \theta'$ such that $\theta \notin \mathcal{E}(\theta')$. Note that $\theta > \theta'$ and $\theta \notin \mathcal{E}(\theta')$ implies that $l(\theta_i = \theta') < l(\theta_i = \theta)$. We argue if $u\left(b|\theta'\right) \geq u\left(b'|\theta'\right)$, then $u\left(b|\theta\right) > u\left(b'|\theta\right)$, where the reference to market m in the function u has been suppressed for simplicity. Writing the utility that type θ' enjoys from submitting bid b, we obtain

$$u\left(b|\theta'\right) = u\left(b'|\theta'\right) + \Pr\left(Y\left(b,b'\right)|\theta'\right)\mathbb{E}\left[V - P|Y\left(b,b'\right),\theta'\right].$$

Observing that

$$\begin{aligned}\Pr\left(Y\left(b,b'\right)|\theta'\right)\mathbb{E}\left[V - P|Y\left(b,b'\right),\theta'\right] = {} & \Pr\left(V=0|\theta'\right)\left(Y\left(b,b'\right)|0\right)\left(\mathbb{E}\left[0 - P|Y\left(b,b'\right),\theta'\right]\right)\\ & + \Pr\left(V=1|\theta'\right)\left(Y\left(b,b'\right)|1\right)\left(\mathbb{E}\left[1 - P|Y\left(b,b'\right),\theta'\right]\right)\end{aligned}$$

we obtain

$$\begin{aligned}u(b|\theta') - u(b'|\theta') = {} & \Pr(V=1|\theta')\Pr(Y(b,b')|1)(1 - \mathbb{E}[P|Y(b,b'),1]) + \\ & \Pr(V=0|\theta')\Pr(Y(b,b')|0)(0 - \mathbb{E}[P|Y(b,b'),0])) \geq 0.\end{aligned}$$

Note that $u(b|\theta') - u(b'|\theta') \geq 0$ and $\Pr(Y(b,b')) > 0$ together imply that $\Pr(Y(b,b')|1)(1 - \mathbb{E}[P|Y(b,b'),1]) > 0$. Because $l(\theta_i = \theta') < l(\theta_i = \theta)$ implies that $\frac{\Pr(1|\theta)}{\Pr(0|\theta)} > \frac{\Pr(1|\theta')}{\Pr(0|\theta')}$, and because $\Pr(Y(b,b')|1)(1 - \mathbb{E}[P|Y(b,b'),1]) > 0$, we obtain the following inequality

$$\Pr(1|\theta)\Pr(Y(b,b')|1)(1 - \mathbb{E}[P|Y(b,b'),1]) + \Pr(0|\theta)\Pr(Y(b,b')|0)(0 - \mathbb{E}[P|Y(b,b'),0])) > 0.$$

However, this implies that $u(b|\theta) - u(b'|\theta) > 0$.

Suppose $\theta^* \in [0,1]$ and $b^* \in [0,1]$. The argument above immediately implies that if $H([0,\theta^*]\times[b^*,1]) > 0$, then $H((\theta^*,1]\times[0,b^*)) = 0$. The conclusion of the lemma then follows directly from Lemma 6 in Pesendorfer and Swinkels (1997).

The fact that if $b(\theta)$ is increasing over an interval (θ',θ''), then

$$b(\theta) \quad = \quad \mathbb{E}\left[V|Y_s^{n-1}(k_s) = \theta, \theta_i = \theta\right]$$

for almost every $\theta \in (\theta',\theta'')$ follows immediately from Pesendorfer and Swinkels (1997). An explicit calculation shows

$$\mathbb{E}\left[V|Y_s^{n-1}(k_s) = \theta, \theta_i = \theta\right] \quad = \quad \frac{l\left(Y_s^{n-1}(k_s) = \theta, \theta_i = \theta\right)}{1 + l\left(Y_s^{n-1}(k_s) = \theta, \theta_i = \theta\right)} \tag{H.1}$$

where

$$l\left(Y_s^{n-1}(k_s) = \theta, \theta_i = \theta\right) = l(\theta_i = \theta)^2 \left(\frac{1-\bar{F}_s^n(\theta|1)}{1-\bar{F}_s^n(\theta|0)}\right)^{n-k_s-1} \left(\frac{\bar{F}_s^n(\theta|1)}{\bar{F}_s^n(\theta|0)}\right)^{k_s-1}.$$

□

Given a pooling bid b_p^n, let $\theta_p^n = \sup\left\{\theta : b^n(\theta) = b_p^n\right\}$, $\underline{\theta}_p^n = \inf\left\{\theta : b^n(\theta) = b_p^n\right\}$, $\lim \theta_p^n = \theta_p$, and $\lim \underline{\theta}_p^n = \underline{\theta}_p$. Let the random variables L^n, G^n, and $X^n = L^n + G^n$ denote the number of losers, number of winners (or the number of objects left for the bidders that submit a bid equal to b_p^n), and number of bidders that submit a bid equal to b_p^n, respectively. Let $\bar{L}^n = \mathbb{E}\left[L^n|P^n = b_p^n, v\right]$, $\bar{G}^n = \mathbb{E}\left[G^n|P^n = b_p^n, v\right]$ and $\bar{X}^n = \bar{L}^n + \bar{G}^n$. Given these definitions, $\Pr\left[b_p^n\ lose|P^n = b_p^n, v\right] = \mathbb{E}\left[\frac{L^n}{X^n}|P^n = b_p^n, v\right]$ and $\Pr\left[b_p^n\ win|P^n = b_p^n, v\right] = \mathbb{E}\left[\frac{G^n}{X^n}|P^n = b_p^n, v\right]$. For any type θ that submits the pooling bid,

$$\Pr(L^n = i|Y_s^n(k_s+1) = \theta, v) = bi\left(i; n-1-k_s, \frac{F_s^n\left(\left[\underline{\theta}_p^n, \theta\right]|v\right)}{1-\bar{F}_s^n\left(\theta|v\right)}\right)$$

and

$$\Pr(X^n = i|Y_s^n(k_s+1) = \theta, v) = bi\left(i; n-1-k_s, \frac{F_s^n\left(\left[\theta, \theta_p^n\right]|v\right)}{\bar{F}_s^n\left(\theta|v\right)}\right).$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[L^n|Y_s^n(k_s+1) = \theta, v\right] &= n\frac{F_s^n\left(\left[\underline{\theta}_p^n, \theta\right]|v\right)}{1-\bar{F}_s^n\left(\theta|v\right)}\left(1-\kappa_s-\frac{1}{n}\right)\\
\mathbb{E}\left[X^n|Y_s^n(k_s+1) = \theta, v\right] &= n\kappa_s\frac{F_s^n\left(\left[\theta, \theta_p^n\right]|v\right)}{\bar{F}_s^n\left(\theta|v\right)}\\
\bar{L}^n &= \int_{\underline{\theta}_p^n}^{\theta_p^n} n\frac{F_s^n\left(\left[\underline{\theta}_p^n, \theta\right]|v\right)}{1-\bar{F}_s^n\left(\theta|v\right)}\left(1-\kappa_s-\frac{1}{n}\right)\Pr\left(Y_s^n(k_s+1) = \theta|v\right)d\theta\\
\bar{X}^n &= \int_{\underline{\theta}_p^n}^{\theta_p^n} n\kappa_s\frac{F_s^n\left(\left[\theta, \theta_p^n\right]|v\right)}{\bar{F}_s^n\left(\theta|v\right)}\Pr\left(Y_s^n(k_s+1) = \theta|v\right)d\theta.
\end{aligned}$$

Lemma H.1. *Suppose* $\lim \Pr\left(P^n = b_p^n | V = v\right) > 0$. *Along any subsequence where the mentioned limits exist, we have the following:*

i. If $\lim \sqrt{n}\left|F_s^n\left(\theta^n(v)|V=v\right) - F_s^n\left(\underline{\theta}_p^n|V=v\right)\right| \to \infty$, *then*

$$0 < \lim \frac{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|V=v\right)}{F_s^n\left(\left[\underline{\theta}_p^n, \theta^n(v)\right]|V=v\right)} \Pr\left(b_p^n \textit{ lose}|P^n = b_p^n, V=v\right) < \infty.$$

ii. If $\lim \sqrt{n}\left|F_s^n\left(\theta^n(v)|V=v\right) - F_s^n\left(\underline{\theta}_p^n|V=v\right)\right| < \infty$, *then*

$$0 < \lim \sqrt{n} F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|V=v\right) \Pr\left(b_p^n \textit{ lose}|P^n = b_p^n, V=v\right) < \infty.$$

iii. If $\lim \sqrt{n}\left|F_s^n\left(\theta^n(v)|V=v\right) - F_s^n\left(\theta_p^n|V=v\right)\right| \to \infty$, *then*

$$0 < \lim \frac{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|V=v\right)}{F_s^n\left(\left[\theta^n(v), \theta_p^n\right]|V=v\right)} \Pr\left(b_p^n \textit{ win}|P^n = b_p^n, V=v\right) < \infty$$

iv. If $\lim \sqrt{n}\left|F_s^n\left(\theta^n(v)|V=v\right) - F_s^n\left(\theta_p^n|V=v\right)\right| < \infty$, *then*

$$0 < \lim \sqrt{n} F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|V=v\right) \Pr\left(b_p^n \textit{ win}|P^n = b_p^n, V=v\right) < \infty$$

Proof. Further below we argue that the expected number of losers at the pooling bid satisfies $0 < \liminf \frac{\bar{L}^n}{\sqrt{n}} \leq \limsup \frac{\bar{L}^n}{\sqrt{n}} < \infty$ if $\lim \sqrt{n}\left|F_s^n\left(\theta^n(v)|V=v\right) - F_s^n\left(\underline{\theta}_p^n|V=v\right)\right| < \infty$, and satisfies

$$0 < \liminf \frac{\bar{L}^n}{nF_s^n\left(\left[\underline{\theta}_p^n, \theta^n(v)\right]|V=v\right)} \leq \limsup \frac{\bar{L}^n}{nF_s^n\left(\left[\underline{\theta}_p^n, \theta^n(v)\right]|V=v\right)} \leq 1$$

if

$$\lim \sqrt{n}\left|F_s^n\left(\theta^n(v)|V=v\right) - F_s^n\left(\underline{\theta}_p^n|V=v\right)\right| \to \infty.$$

We will prove items i and ii that refer to $\Pr\left(b_p^n \textit{ lose}|P^n = b_p^n, v\right)$ using these bounds for $\bar{L}^n$. We begin by proving the lower bounds in items i and ii. Note that $\Pr\left(L^n \geq \bar{L}^n - 1|P^n = b_p^n, v\right) \geq 1/2$.[35]

$$\begin{aligned}
\Pr\left(b_p^n \textit{ lose}|P^n = b_p^n, v\right) &\geq \mathbb{E}\left[\frac{L^n}{X}|L^n \geq \bar{L}^n - 1, P^n = b_p^n, v\right] \Pr\left(L^n \geq \bar{L}^n - 1|P^n = b_p^n, v\right) \\
&\geq \mathbb{E}\left[\frac{\bar{L}^n - 1}{X}|L^n \geq \bar{L}^n - 1, P^n = b_p^n, v\right] \frac{1}{2} \\
&\geq \frac{\left(\bar{L}^n - 1\right)/2}{\mathbb{E}\left[X^n|L^n \geq \bar{L}^n - 1, P^n = b_p^n, v\right]} \text{ (by Jensen's Inequality)}
\end{aligned}$$

Note that $\mathbb{E}\left[X^n|L^n \geq \bar{L}^n - 1, P^n = b_p^n, v\right] \Pr\left(L^n \geq \bar{L}^n - 1, P^n = b_p^n|v\right) \leq \mathbf{E}\left[X^n|v\right] = nF_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|v\right)$.

[35] Conditional on $Y_s^n(k_s+1) = \theta \in \left[\underline{\theta}_p^n, \theta_p^n\right]$ and $V = v$, the number of losers L^n is a binomial random variable. The median of the binomial differs from the mean by at most one. Therefore, $\Pr\left(L^n \geq \mathbf{E}[L^n|Y_s^n(k_s+1) = \theta, v] - 1|Y_s^n(k_s+1) = \theta, v\right) \geq 1/2$. In turn, this implies that $\Pr\left(L^n \geq \bar{L}^n - 1|P^n = b_p^n, v\right) \geq 1/2$.

Therefore,

$$\Pr\left(b_p^n\ lose|P^n = b_p^n, v\right) \geq \frac{(\bar{L}^n - 1)}{nF_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|v\right)} \frac{\Pr\left(L^n \geq \bar{L}^n - 1|P^n = b_p^n, v\right)\Pr\left(P^n = b_p^n|v\right)}{2}$$

$$\Pr\left(b_p^n\ lose|P^n = b_p^n, v\right) nF_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|v\right) \geq (\bar{L}^n - 1)\frac{\Pr\left(P^n = b_p^n|v\right)}{4}$$

Taking limits and substituting

$$0 < \liminf \frac{\bar{L}^n - 1}{\sqrt{n}} < \limsup \frac{\bar{L}^n - 1}{\sqrt{n}} < \infty$$

if

$$\lim \sqrt{n}\left|F_s^n\left(\theta^n(v)|V = v\right) - F_s^n\left(\underline{\theta}_p^n|V = v\right)\right| < \infty;$$

and

$$0 < \liminf \frac{\bar{L}^n - 1}{nF_s^n\left(\left[\underline{\theta}_p^n, \theta^n(v)\right]|V = v\right)} \leq \limsup \frac{\bar{L}^n - 1}{nF_s^n\left(\left[\underline{\theta}_p^n, \theta^n(v)\right]|V = v\right)} \leq 1$$

if

$$\lim \sqrt{n}\left|F_s^n\left(\theta^n(v)|V = v\right) - F_s^n\left(\underline{\theta}_p^n|V = v\right)\right| \to \infty$$

delivers the lower bounds in items i and ii.

We now establish the upper bounds in items i and ii. If $\lim \sqrt{n}F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|v\right) \in (0, \infty)$, then $\lim \sqrt{n}\left|F_s^n\left(\theta^n(v)|V = v\right) - F_s^n\left(\underline{\theta}_p^n|V = v\right)\right| < \infty$ (because $\lim \Pr\left(P^n = b_p^n|V = v\right) > 0$) and the upper bound in item ii is trivially satisfied. Suppose $\lim \sqrt{n}F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|v\right) = \infty$. Pick $\delta \in (0, 1)$ and let $\bar{Y}^n = nF_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|v\right)$. Then

$$\Pr\left[b_p^n\ lose|P^n = b_p^n, v\right] \leq \frac{\mathbb{E}[L^n|X^n > (1-\delta)\bar{Y}^n, P^n = b_p^n, v]\Pr\left(X^n > (1-\delta)\bar{Y}^n|P^n = b_p^n, v\right)}{(1-\delta)\bar{Y}^n} + \Pr\left(X^n \leq \bar{Y}^n(1-\delta)|P^n = b_p^n, v\right).$$

However, $\mathbb{E}[L^n|X^n > (1-\delta)\bar{Y}^n, P^n = b_p^n, v]\Pr\left(X^n > (1-\delta)\bar{Y}^n|P^n = b_p^n, v\right) \leq \bar{L}^n$. Therefore,

$$\frac{\Pr\left[b_p^n\ lose|P^n = b_p^n, v\right]\bar{Y}^n}{\bar{L}^n} \leq \frac{1}{1-\delta} + \frac{\bar{Y}^n}{\bar{L}^n}\Pr\left(X^n \leq \bar{Y}^n(1-\delta)|P^n = b_p^n, v\right)$$

Chernoff's inequality implies that $\lim \Pr\left(X^n \leq (1-\delta)\bar{Y}^n|v\right) < \exp\left(-\frac{\delta^2\bar{Y}^n}{3}\right)$ and hence

$$\Pr\left(X^n \leq \bar{Y}^n(1-\delta)|P^n = b_p^n, v\right) \leq \frac{\exp\left(-\frac{\delta^2 Y^n}{3}\right)}{\Pr\left(P^n = b_p^n|v\right)}.$$

Therefore, $\lim \frac{\Pr\left[b_p^n\ lose|P^n = b_p^n, v\right]\bar{Y}^n}{\bar{L}^n} \leq \frac{1}{1-\delta}$. Substituting for the number of losers $\bar{L}^n$ now delivers the upper bounds in items i and ii.

We now show that $0 < \liminf \frac{\bar{L}^n}{\sqrt{n}} \leq \limsup \frac{\bar{L}^n}{\sqrt{n}} < \infty$ if $\lim \sqrt{n}\left|F_s^n\left(\theta^n(v)|V = v\right) - F_s^n\left(\underline{\theta}_p^n|V = v\right)\right| <$

∞, and satisfies

$$0 < \liminf \frac{\bar{L}^n}{nF_s^n\left(\left[\underline{\theta}_p^n, \theta^n(v)\right] | V = v\right)} \leq \limsup \frac{\bar{L}^n}{nF_s^n\left(\left[\underline{\theta}_p^n, \theta^n(v)\right] | V = v\right)} \leq 1$$

if

$$\lim \sqrt{n}\left|F_s^n\left(\theta^n(v)|V=v\right) - F_s^n\left(\underline{\theta}_p^n|V=v\right)\right| \to \infty.$$

Pick any $\theta^n \in \left[\underline{\theta}_p^n, \theta_p^n\right]$ and let $a(\theta^n) := \bar{F}_s^n\left(\theta^n(v)|v\right) - \bar{F}_s^n\left(\theta^n|v\right) = \kappa_s - \bar{F}_s^n\left(\theta^n|v\right)$. Recall that $\Pr(L^n = i|Y_s^n(k_s+1) = \theta^n, v) = bi\left(i; n-1-k_s, p^n\right)$ and $\mathbb{E}\left[L^n|Y_s^n(k_s+1) = \theta^n, v\right] = np(a(\theta^n))\left(1-\kappa_s-\frac{1}{n}\right)$ where $p(a(\theta^n)) = \frac{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)-\kappa_s+a(\theta^n)}{1-\kappa_s+a(\theta^n)}$. Calculating the number of losers we find $\bar{L}^n = -\left(1-\kappa_s-\frac{1}{n}\right)\int_{\underline{a}^n}^{\bar{a}^n} np(a)d\Lambda(a)$ where $\underline{a}^n = a(\underline{\theta}_p^n)$, $\bar{a}^n = a(\theta_p^n)$, and $\Lambda(a) := \Pr\left(F_s^n\left(Y_s^n(k_s+1)|v\right) - \kappa_s \geq a|P = b_p^n, v\right)$. Integrating by parts and substituting $p(\underline{a}^n) = 0$, $\Lambda(\bar{a}^n) = 0$, and $p'(a) = \frac{\left(1-\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)\right)}{(1-\kappa_s+a)^2} = \frac{1-\kappa_s+\underline{a}^n}{(1-\kappa_s+a)^2}$ delivers

$$\bar{L}^n/n = \left(1-\kappa_s-\frac{1}{n}\right)\left(1-\kappa_s+\underline{a}^n\right)\int_{\underline{a}^n}^{\bar{a}^n} \frac{\Lambda(a)}{(1-\kappa_s+a)^2}da.$$

Hence,

$$C^n\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da \leq \bar{L}^n/n \leq \frac{1-\kappa_s}{1-\kappa_s+\underline{a}^n}\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da \leq \int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da.$$

where $C^n = \frac{(1-\kappa_s)(1-\kappa_s+\underline{a}^n)}{(1-\kappa_s+\bar{a}^n)^2}$.

Pick any $\epsilon > 0$ and let a_*^n be such that $\Pr(F_s^n\left(Y_s^n(k_s+1)|v\right) - \kappa_s \geq a^n|P^n = b_p^n, v) = 1-\epsilon$. The central limit theorem implies that $\lim\sqrt{n}a_*^n \in (0,\infty)$. Moreover, $\lim\sqrt{n}(a^n - \underline{a}^n) > 0$ because $\Pr(F_s^n\left(Y_s^n(k_s+1)|v\right) - \kappa_s \leq a^n|P^n = b_p^n, v) = \epsilon$ for each n. Therefore,

$$\begin{aligned}
\int_{\underline{a}^n}^{a_*^n}\Lambda(a)da \leq \int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da &\leq \max\{-\underline{a}^n, 0\} + \int_0^{\bar{a}^n}\Lambda(a)da \\
\int_{\underline{a}}^{a_*^n}(1-\epsilon)da \leq \int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da &\leq \max\{-\underline{a}^n, 0\} + \frac{\int_0^{\bar{a}^n} e^{-\frac{a^2n}{2}}d\theta}{\Pr\left(P^n = b_p^n|v\right)} \text{ (by Chernoff's inequality)} \\
(1-\epsilon)\left(a_*^n - \underline{a}^n\right) \leq \int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da &= \max\{-\underline{a}^n, 0\} + \frac{\sqrt{2}erf\left(\sqrt{n}\bar{a}\right)}{\Pr\left(P^n = b_p^n|v\right)\sqrt{\pi n}} \\
&\leq \max\{-\underline{a}^n, 0\} + \frac{1}{\Pr\left(P^n = b_p^n|v\right)\sqrt{2\pi n}}
\end{aligned}$$

where $erf(x) = \frac{1}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \in [0, 1/2]$ is the error function.

Note $-\underline{a}^n = F_s^n\left(\theta^n(v)|V=v\right) - F_s^n\left(\underline{\theta}_p^n|V=v\right)$. Suppose that $-\lim\sqrt{n}\underline{a}^n < \infty$. If $-\lim\sqrt{n}\underline{a}^n = \delta_1 < \infty$, then $\lim\sqrt{n}(a_*^n - \underline{a}^n) = \delta \in (0,\infty)$. The fact that $\frac{\bar{L}^n}{\sqrt{n}} \in \left(\sqrt{n}C^n\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da, \sqrt{n}\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da\right)$ and the bounds for $\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da$ together imply that

$$\begin{aligned}
C^n\left((1-\epsilon)\sqrt{n}(a_*^n - \underline{a}^n)\right) \leq \frac{\bar{L}^n}{\sqrt{n}} &\leq \max\{\sqrt{n}\underline{a}^n, 0\} + \frac{1}{\Pr\left(P^n = b_p^n|v\right)\sqrt{2\pi}} \\
0 < (1-\epsilon)C\delta \leq \liminf\frac{\bar{L}^n}{\sqrt{n}} \leq \limsup\frac{\bar{L}^n}{\sqrt{n}} &\leq \max\{\delta_1, 0\} + \frac{1}{\Pr\left(P^n = b_p^n|v\right)\sqrt{2\pi}} < \infty
\end{aligned}$$

where $C = \liminf C^n$.

If $-\lim \sqrt{n}\underline{a}^n = \infty$, then $L^n \in \left(n\sqrt{n}C^n \int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)da, n\int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)da\right)$ and the bounds for $\int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)da$ together imply that

$$C^n\left(\frac{(1-\epsilon)\,n\,(a_*^n - \underline{a}^n)}{-n\underline{a}^n}\right) \le \frac{\bar{L}^n}{-n\underline{a}^n} \le 1 - \frac{1}{\Pr\left(P^n = b_p^n|v\right)\underline{a}^n\sqrt{2\pi n}}$$

$$\lim C^n\left((1-\epsilon)\left(\frac{\sqrt{n}a_*^n}{-n\underline{a}^n}+1\right)\right) \le \liminf \frac{\bar{L}^n}{-n\underline{a}^n} \le \limsup \frac{\bar{L}^n}{-n\underline{a}^n} \le 1$$

$$0 < C\,(1-\epsilon) \le \liminf \frac{\bar{L}^n}{-n\underline{a}^n} \le \limsup \frac{\bar{L}^n}{-n\underline{a}^n} \le 1.$$

□

Proof of A.4. As before, let X^n denote the random variable which is equal to the number of bidders in the interval $\left[\underline{\theta}_p^n, \theta_p^n\right]$. Redefine L^n to denote the random variable which is equal to the number of losers with signals that exceed $\underline{\theta}_p^n$. Note that $\mathbb{E}\left[L^n|Y_s^n(k+1) \ge \underline{\theta}_p^n, V=v\right] = \mathbb{E}\left[L^n|L^n \ge 1, V=v\right]$. Pick a $\delta > 0$, and let $d^n = (1-\delta)k_s\frac{F_s^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right)}{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)}$ and observe that $\lim \frac{d^n}{\sqrt{n}} > 0$. We will show

$$\lim \frac{\mathbb{E}\left[\frac{L^n}{X^n}|L^n \in [1,d^n], V=0\right]}{\frac{\bar{F}_s^n\left(\underline{\theta}_p^n|V=v\right)\left(1-\bar{F}_s\left(\underline{\theta}_p|V=v\right)\right)}{nF_s^n\left([\underline{\theta}_p^n,\theta_p^n]|V=v\right)\left(\kappa_s-\bar{F}_s\left(\underline{\theta}_p|V=v\right)\right)}} = 1$$

and $\lim \frac{\Pr\left(b_p^n\ loses|P^n=b_p^n,V=v\right)}{\mathbb{E}\left[\frac{L^n}{X^n}|L^n\in[1,d^n],V=0\right]} = 1$.

Step 1. We will first show $\lim \frac{\mathbb{E}[L^n|L^n\in[1,d^n],v]}{a^n} = 1$, where

$$a^n = \frac{\kappa_s\left(1-\bar{F}_s^n\left(\underline{\theta}_p^n|V=v\right)\right)}{\kappa_s - \bar{F}_s^n\left(\underline{\theta}_p^n|V=v\right)}.$$

Note

$$\lim \mathbb{E}\left[L^n|L^n \in [1,d^n], v\right] = \frac{\sum_{i=1}^{d^n} ibi(k_s+i,n;p^n)}{\sum_{i=1}^{d^n} bi(k_s+i,n;p^n)}$$

where $p^n = \bar{F}_s^n(\underline{\theta}_p^n|v)$. Observe that

$$\frac{bi(k+i,n;p^n)}{bi(k+i,n;\kappa_s)}bi(k+i,n;\kappa_s) = bi(k+i,n;\kappa_s)\left(\frac{p^n}{\kappa_s}\right)^{k_s}\left(\frac{1-p^n}{1-\kappa_s}\right)^{n-k_s}\left(\frac{p^n(1-\kappa_s)}{\kappa_s(1-p^n)}\right)^i.$$

Therefore

$$\mathbb{E}\left[L^n|L^n \in [1,d^n], v_i\right] = \frac{\sum_{i=1}^{d^n} ir(n)^i bi(k_s+i,n;\kappa_s)}{\sum_{i=1}^{d^n} r(n)^i bi(k_s+i,n;\kappa_s)}$$

where $r(n) = \frac{p^n(1-\kappa_s)}{\kappa_s(1-p^n)} < 1$. Pick any $J < d^n$. For each $i < J$,

$$(1-\epsilon^n)\phi\left(\frac{J}{\sqrt{n\,(1-\kappa_s)\,\kappa_s}}\right) \le \sqrt{n\,(1-\kappa_s)\,\kappa_s}bi(k+i,n;\kappa_s) \le (1+\epsilon^n)\phi(0)$$

by the local limit theorem (Proposition A.4). Hence,

$$(1-\epsilon^n)\frac{\phi\left(\frac{J}{\sqrt{n}}\right)\sum_{i=1}^{J} ir(n)^i}{\phi\left(0\right)\sum_{i=1}^{d^n} r(n)^i} \leq \frac{\sum_{i=1}^{d^n} ir^i\sqrt{n}bi(k+i,n;\kappa_s)}{\sum_{i=1}^{d^n} r^i\sqrt{n}bi(k+i,n;\kappa_s)} \leq \frac{\phi(0)}{\phi\left(\frac{J}{\sqrt{n}}\right)}\frac{\sum_{i=1}^{d^n} ir(n)^i}{\sum_{i=1}^{J} r(n)^i}(1+\epsilon^n).$$

Evaluating the geometric series we find

$$\frac{\phi\left(\frac{J}{\sqrt{n}}\right)(1-\epsilon^n)}{\phi\left(0\right)\left(1-r\left(n\right)^{d^n}\right)}\left(\frac{1-r(n)^J}{1-r\left(n\right)}-Jr\left(n\right)^J\right) \leq Q \leq \frac{\phi(0)\left(1+\epsilon^n\right)}{\phi\left(\frac{J}{\sqrt{n}}\right)\left(1-r\left(n\right)^J\right)}\left(\frac{1-r(n)^{d^n}}{1-r\left(n\right)}-d^n r\left(n\right)^{d^n}\right)$$
$$\frac{\phi\left(\frac{J}{\sqrt{n}}\right)(1-\epsilon^n)}{\phi\left(0\right)\left(1-r\left(n\right)^{d^n}\right)}\left(\frac{1-r(n)^J}{1-r\left(n\right)}-Jr\left(n\right)^J\right) \leq Q \leq \frac{\phi(0)\left(1+\epsilon^n\right)}{\phi\left(\frac{J}{\sqrt{n}}\right)\left(1-r\left(n\right)^J\right)}$$

where $Q=\mathbb{E}\left[L^n|L^n\in\left[1,d^n\right],v\right]$.

Case 1. $\bar{F}(\underline{\theta}_p|v)<\kappa_s$. In this case, $\lim r(n)=r<1$. Picking $J=n^{1/4}<d^n$ and taking the limit as $n\to\infty$ we find

$$\lim\mathbb{E}\left[L^n|L^n\in\left[1,d^n\right],v_i\right]=\frac{1}{1-r}=\frac{\kappa_s\left(1-\bar{F}_s\left(\underline{\theta}_p|V=v\right)\right)}{\kappa_s-\bar{F}_s\left(\underline{\theta}_p|V=v\right)}=\lim a^n$$

Case 2. $\overline{F}_s\left(\underline{\theta}_p|v_i\right)=\kappa_s$. In this case $r(n)<1$ for all n sufficiently large but $\lim r(n)=1$. Note that

$$\lim\frac{1-r(n)}{1/a^n}=1.$$

For any constant m, $ma^n<d^n$ for sufficiently large n because $d^n/a^n\to\infty$. Substituting $1/a^n$ for $1-r(n)$ and setting $J=ma^n$ for any arbitrary m we find

$$\frac{\phi\left(\frac{ma^n}{\sqrt{n}}\right)}{\phi(0)}\frac{a^n\left(1-\left(1-1/a^n\right)^{ma^n}\right)-ma^n\left(1-1/a^n\right)^{ma^n}}{1-\left(1-1/a^n\right)^{d^n}}\frac{1-\epsilon^n}{a^n}\leq X\leq\frac{\phi(0)}{\phi\left(\frac{ma^n}{\sqrt{n}}\right)}a^n\frac{1+\epsilon^n}{a^n}$$
$$\frac{\phi\left(\frac{ma^n}{\sqrt{n}}\right)}{\phi(0)}\frac{\left(1-\left(1-1/a^n\right)^{ma^n}\right)-m\left(1-1/a^n\right)^{ma^n}}{1-\left(1-1/a^n\right)^{d^n}}(1-\epsilon^n)\leq X\leq\frac{\phi(0)}{\phi\left(\frac{ma^n}{\sqrt{n}}\right)}(1+\epsilon^n)$$

where $X=\frac{\mathbb{E}[L^n|L^n\in[1,d^n],v_i]}{a^n}$. Taking the limit as $n\to\infty$ and noting that $a^n\to\infty$, $a^n/\sqrt{n}\to0$ and $d^n/a^n\to\infty$ we obtain $\left(1-1/a^n\right)^{ma^n}\to\exp\left(-m\right)$, $\phi\left(\frac{ma^n}{\sqrt{n}}\right)\to\phi\left(0\right)$, and $\left(1-1/a^n\right)^{d^n}\to0$. Therefore

$$1-\exp\left(-m\right)-\exp\left(-m\right)m\leq\lim\frac{\mathbb{E}\left[L^n|L^n\in\left[1,d^n\right],v_i\right]}{a^n}\leq1$$

As m is arbitrary, taking the limit as $m\to\infty$ we find $\frac{\mathbb{E}[L^n|L^n\in[1,d^n],v_1]}{a^n}\to1$.

Step 2. We show $\Pr\left(L^n>d^n|L^n\geq1,v\right)\leq A\exp\left(-\frac{d^n}{a^n}\right)\to0$ and

$$\Pr\left[Y^n(k+1)>\theta_p^n|Y^n(k+1)>\underline{\theta}_p^n,v\right]\leq A\exp\left(-\frac{d^n}{a^n}\right)\to0$$

where A is an arbitrary positive constant.

Following the procedure from the previous step, we find

$$\Pr\left(L^n > d^n | L^n \geq 1, v\right) = \frac{\sum_{i=d^n}^{n-k} bi(k+i,n;\kappa)}{\sum_{i=1}^{n-k} bi(k+i,n;\kappa)}$$
$$= \frac{r^{d^n}(1-\epsilon_1^n)\left(a^n + \frac{\left(1-\frac{1}{a^n}\right)^n}{a^n} - \left(1-\frac{1}{a^n}\right)^n\right)}{(1-\epsilon_2^n)\left(a^n + \frac{\left(1-\frac{1}{a^n}\right)^n}{a^n} - \left(1-\frac{1}{a^n}\right)^n\right)} \leq A\exp\left(-\frac{d^n}{a^n}\right)$$

where last inequality is a consequence of the fact that $\left(1-\frac{1}{a^n}\right)^{d^n}$ is of the order of $\exp\left(-\frac{d^n}{a^n}\right)$. Also, we have

$$\Pr\left[Y^n(k+1) > \theta_p^n | Y^n(k+1) > \underline{\theta}_p^n, v\right] =$$
$$\Pr\left(L^n \in [1,d^n] | L^n > 1, v\right) \Pr\left(X^n < L^n | L^n \in [1,d^n], L^n > 1, v\right)$$
$$+ \Pr\left(L^n > d^n | L^n > 1, v\right) \Pr\left(X^n < L^n | L^n > d^n, L^n > 1, v\right).$$

Consequently,

$$\Pr\left[Y^n(k+1) > \theta_p^n | Y^n(k+1) > \theta_p^n, v\right] \leq \Pr\left(L^n > d^n | L^n > 1, v\right) + \Pr\left(X^n < L^n | L^n \in [1,d^n], L^n > 1, v\right)$$
$$\leq \frac{\sum_{i=d^n}^{n-k} bi(k+i,n;\kappa)}{\sum_{i=1}^{n-k} bi(k+i,n;\kappa)} + \exp\left(-\frac{\delta^2}{2}d^n\right)$$
$$\leq A\exp\left(-\frac{d^n}{a^n}\right) + \exp\left(-\frac{\delta^2}{2}d^n\right)$$
$$\leq A\exp\left(-\frac{d^n}{a^n}\right)$$

where in the last inequality we use the fact that $A\exp\left(-\frac{d^n}{a^n}\right) \geq \exp\left(-\frac{\delta^2}{2}d^n\right)$ and redefine the constant A without changing the order of the term.

Step 3. We now show

$$\frac{\overline{F}_s^n\left(\underline{\theta}_p^n | v\right)}{(1+\delta)F^n\left([\underline{\theta}_p^n, \theta_p^n] | v\right)} \frac{\mathbb{E}\left[L^n | L^n \in [1,d^n], v\right]}{k_s} \leq \Pr\left(b^p\ loses | L^n \geq 1, v\right) \leq$$
$$\frac{\overline{F}_s^n\left(\underline{\theta}_p^n | v\right)}{(1-\delta)F^n\left([\underline{\theta}_p^n, \theta_p^n] | v\right)} \frac{\mathbb{E}\left[L^n | L^n \in [1,d^n], v\right]}{k_s} + A\exp\left(-\frac{d^n}{a^n}\right)$$

We first give a lower bound for the probability of losing:

$$\Pr\left(b^p\ loses | L^n \geq 1, v\right) \geq \mathbb{E}\left[\min\left[\frac{L^n}{X^n}, 1\right] | L^n \in [1,d^n], v\right] \Pr\left(L^n \in [1,d^n] | L^n \geq 1, v\right)$$

Note that $\Pr\left(L^n \in [1,d^n] | L^n \geq 1, v\right) \to 1$, thus

$$\Pr\left(b^p\ loses | L^n \geq 1, v\right) \geq \mathbb{E}\left[\min\left[\frac{L^n}{X^n}, 1\right] | L^n \in [1,d^n], v\right] (1-\delta_1)$$

where δ_1 is an arbitrarily small constant. The facts that $\min\left[\frac{L^n}{X^n}, 1\right]$ is a concave function of X^n

and Jensen's inequality together imply that

$$\mathbb{E}\left[\min\left[\frac{L^n}{X^n},1\right]|L^n\in[1,d^n],v\right]\geq\mathbb{E}\left[\min\left[\frac{L^n}{\mathbb{E}\left[X^n|L^n,v\right]},1\right]|L^n\in[1,d^n],v\right].$$

By definition $\mathbb{E}\left[X^n|L^n,v_i\right]>d^n$, therefore

$$\begin{aligned}\mathbb{E}\left[\min\left[\frac{L^n}{\mathbb{E}\left[X^n|L^n,v_i\right]},1\right]|L^n\in[1,d^n],v\right]&=\mathbb{E}\left[\frac{L^n}{\mathbb{E}\left[X^n|L^n,v_i\right]}|L^n\in[1,d^n],v\right]\\&=\frac{\overline{F}_s^n\left(\underline{\theta}_p^n|v\right)}{F_s^n\left(\left[\underline{\theta}_p^n,\theta_p^n\right]|v\right)}\mathbb{E}\left[\frac{L^n}{L^n+k_s}|L^n\in[1,d^n],v\right]\end{aligned}$$

Noticing that $\frac{L^n}{L^n+k}$ is a concave function of L and applying Jensen's inequality implies that

$$\begin{aligned}\mathbb{E}\left[\min\left[\frac{L^n}{X^n},1\right]|L^n\in[1,d^n],v\right]&\geq\frac{\overline{F}_s^n\left(\underline{\theta}_p^n|v\right)}{F_s^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right)}\frac{\frac{\mathbb{E}[L^n|L^n\in[1,d^n],v]}{k_s}}{\frac{\mathbb{E}[L^n|L^n\in[1,d^n],v]}{k_s}+1}\\&\geq\frac{\overline{F}_s^n\left(\underline{\theta}_p^n|v\right)}{F_s^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right)}\frac{\frac{\mathbb{E}[L^n|L^n\in[1,d^n],v]}{k_s}}{1+\delta_2}\end{aligned}$$

where $\delta_2:=\frac{\mathbb{E}[L^n|L^n\in[1,d^n],v]}{k}$ is an arbitrary positive constant. Note that $\frac{\mathbb{E}[L^n|L^n\in[1,d^n],v]}{k}\to 0$, therefore we can choose δ_2 arbitrarily small for large n. Therefore,

$$\Pr\left(b^p\ loses|L^n\geq 1,v\right)\geq(1-\delta)\frac{\overline{F}_s^n\left(\underline{\theta}_p^n|v\right)}{F_s^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right)k_s}\mathbb{E}\left[L^n|L^n\in[1,d^n],v\right]$$

where $1-\delta=\min\left\{\frac{1}{1+\delta_2},1-\delta_1\right\}$.

We now provide an upper bound for the probability of losing:

$$\begin{aligned}&\Pr\left(b^p\ loses|L^n\geq 1,v\right)\leq\\&\quad\mathbb{E}\left[\min\left[\frac{L^n}{X^n},1\right]|L^n\in[1,d^n],v\right]+\Pr\left(L^n>d^n|L^n\geq 1,v\right)\leq\\&\frac{\overline{F}_s^n\left(\underline{\theta}_p^n|v\right)}{(1-\delta)F_s^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right)}\mathbb{E}\left[\frac{L^n}{L^n+k_s}|L^n\in[1,d^n],v\right]+\Pr\left(L^n>d^n|L^n\geq 1,v\right)+\exp\left(-\frac{\delta^2d^n}{2+\delta}\right)\leq\\&\quad\frac{\overline{F}_s^n\left(\underline{\theta}_p^n|v\right)}{(1-\delta)F_s^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right)}\frac{\mathbb{E}\left[L^n|L^n\in[1,d^n],v\right]}{k_s}+A\exp\left(-\frac{d^n}{a^n}\right)+\exp\left(-\frac{\delta^2d^n}{2+\delta}\right)\leq\\&\qquad\frac{\overline{F}_s^n\left(\underline{\theta}_p^n|v\right)}{(1-\delta)F^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right)}\frac{\mathbb{E}\left[L^n|L^n\in[1,d^n],v\right]}{k_s}+A\exp\left(-\frac{d^n}{a^n}\right).\end{aligned}$$

the first inequality follows because $\mathbb{E}\left[X^n|L^n=i\in[1,d^n],v\right]=(k_s+i)\frac{F_s^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right)}{\overline{F}_s^n\left(\underline{\theta}_p^n|v\right)}$ is less than $(1-\delta)(k_s+i)\frac{F_s^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right)}{\overline{F}_s^n\left(\underline{\theta}_p^n|v\right)}$ with probability $\exp\left(-\frac{\delta^2d^n}{2+\delta}\right)$ by Chernoff's inequality and the second follows because we showed that $\Pr\left(L^n>d^n|L^n\geq 1,v\right)\leq A\exp\left(-\frac{d^n}{a^n}\right)$ in step 2. To obtain the last inequality we use the fact $A\exp\left(-\frac{d^n}{a^n}\right)>\exp\left(-\frac{\delta^2d^n}{2+\delta}\right)$ and redefine the constant A without changing the order of the term. The lemma now follows as $\frac{d^n}{a^n}\exp\left(-\frac{d^n}{a^n}\right)\to 0$ because $d^n/a^n\to\infty$ and because the constants δ are arbitrary. $\square$