Market Selection and the Information Content of Prices*

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Abstract

We study information aggregation when n bidders choose, based on their private information, between two concurrent common-value auctions. There are k_s identical objects on sale through a uniform price auction in market s and there are an additionally k_r objects on auction in market r, which is identical to market s except for a positive reserve price. The reserve price in market r implies that information is not aggregated in this market. Moreover, if the object-to-bidder ratio in market s exceeds a certain cutoff, then information is not aggregated in market s either. Conversely, if the object-to-bidder ratio is less than this cutoff, then information is aggregated in market s as the market grows arbitrarily large. Our results demonstrate how frictions in one market can disrupt information aggregation in a linked, frictionless market because of the pattern of market selection by imperfectly informed bidders.

Keywords: Auctions, large markets, information aggregation.

JEL Codes: C73, D44, D82, D83.

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1. Introduction

Consider a market where k_s identical common-value objects of unknown value are sold to n bidders, each with unit demand. The sale is conducted through a sealed-bid auction where each of the highest k_s bidders receives an object and pays a uniform price equal to the highest losing bid. Each object's common value (V) is equal to one in the good state and zero in the bad state. In such an auction, if each bidder has an independent signal about the unknown value of the object, then the auction's equilibrium price converges to the object's true value as the number of objects and the number of bidders grow arbitrarily large (see Pesendorfer and Swinkels (1997)). Therefore, the auction price reveals the unknown value of the object and thus aggregates all relevant information dispersedly held by the bidders.

Most previous work on auctions takes the distribution of types that bid in the auction as exogenously given.[1] Yet, in many instances, bidders strategically decide whether to trade in a particular market after weighing their alternatives. In other words, the bidder distribution is endogenously determined jointly by the set of available alternatives and the bidders' expectations about the relative attractiveness of these alternatives. Our focus in this paper is an environment where bidders choose, based on their private information, between the auction (market s) and an outside option (market r). This framework allows us to highlight the interplay between self-selection into an auction, bidding behavior in the auction, and the information content of prices.

Market r, which serves as the outside option for market s, is a uniform-price auction with a reserve price $c>0$ where there are an additional $n\kappa_r = k_r$ units of the same object for sale.[2] If the object-to-bidder ratio in market r is sufficiently large, then each bidder can purchase an object at a fixed price equal to $c>0$. In this case, the payoff from choosing the outside option is *exogenously* determined by the reserve price c. Otherwise, the attractiveness of the outside option is *endogenously* determined by the bidders that select market r together with the reserve price c.

Our main result identifies when frictions in market r, resulting from the positive reserve price, disrupts information aggregation also in the frictionless market s. In particular, we show that there is no symmetric equilibrium that aggregates information in either market if the object-to-bidder ratio in market s exceeds a certain cutoff $\bar{\kappa}$. This cutoff depends on the reserve price, the signal structure, and the object-to-bidder ratio in market r. If, on the other hand, the object-to-bidder ratio in market s is less than $\bar{\kappa}$, then

[1] Papers by Lauermann and Wolinsky (2017) and Murto and Valimaki (2014) are notable exceptions.

[2] The reserve price has various interpretations: (1) It is a reserve price set by a single auctioneer, (2) The auction is comprised of k_r nonstrategic sellers and the reservation value (or the cost) for these sellers is equal to c; (3) It is a government/regulator imposed minimum price.

information is aggregated in market s. Importantly, our result implies that information aggregation can fail in both markets under imperfect information even in circumstances where information is aggregated in both markets under complete information. We provide intuition for these findings using an illustrative example further below.

Our equilibrium characterization further elucidates the mechanism that leads to prices that fail to aggregate all available information. For the case where market r is perturbed by a small friction ($c > 0$ but small), we characterize all symmetric equilibria. In every equilibrium, the expected prices are equalized across the two markets in each state. Therefore, from the perspective of a bidder who wins an object with probability one, the state-contingent payoffs are also equalized across markets. The pattern of self-selection is the main force that equalizes prices and therefore state-contingent payoffs. A large disparity in the state-contingent payoffs across the two markets would imply that optimistic bidders select the market with large losses in state 0 that are compensated by large gains in state 1 (i.e., the market with higher payoff variance) while pessimistic bidders select the option with lower payoff variance. In other words, market selection would have a cutoff structure. However, if market selection has a cutoff structure and an auction attracts the type distribution's upper tail (i.e., the more optimistic types), then we show that the price is driven towards the object's value in each state, decreasing the payoff variance. In contrast, if an auction attracts the type distribution's lower tail (i.e., the more pessimistic types), then the price diverges from the object's value in each state, which increases the payoff variance. Thus, an equilibrium is sustained only if types self-select across markets in a way that equalizes state-contingent payoffs (and expected prices) in the two markets. In such equilibria, price diverges from value in both markets and fails to aggregate information.

Previous work on information aggregation mainly focused on homogeneous (or highly correlated) objects that trade in a single centralized, frictionless auction market. However, such a centralized market is an exception rather than the rule. Fragmentation, the disperse trading of the same security in multiple markets, is commonplace: Many stocks listed on the New York Stock Exchange trade concurrently on regional exchanges (see Hasbrouck (1995)). Investors, who participate in a primary treasury bond auction, could purchase a bond with similar cashflow characteristics from the secondary market. Labor markets are linked but also segmented according to industry, geography, and skill. Buyers in the market for aluminum or steel can choose between the London Metal Exchange or the New York Mercantile Exchange. Such fragmented markets and exchanges also differ in structure, rules and regulations. In particular, markets are heterogeneous in terms of the frictions that participants face. The results that we present in this paper suggest that selection into markets can have important implications for the information content of prices, especially when individuals choose between markets that differ in terms of institutional detail and therefore frictions. In particular, we demonstrate how frictions can disrupt information aggregation not only in the market with frictions but also in frictionless, substitute markets because of how imperfectly informed bidders select across markets.

1.1. An Illustrative Example. Recall that bidders choose, based on their private information, between market s where there are $n\kappa_s$ objects on auction and market r where there are an additional $n\kappa_r$ objects on auction. The two markets differ in that there is a positive reserve price c in market r while there is no reserve price in market s. For this example, assume that $\kappa_r+\kappa_s<1$ and further suppose that each bidder receives a private signal that perfectly reveals the value of the object with probability $1-g\geq 0$ and receives an uninformative signal with probability g. A bidder who receives the uninformative signal believes that $V=1$ with probability $1/2$ while a bidder who receives the perfectly revealing signal knows the object's true value.

As a first benchmark suppose that $\kappa_r=0$, i.e., suppose that there is only one active market. In this case, it's innocuous to assume that all bidders participate in market s because a non-participating bidder's payoff is equal to zero in both states. However, if all bidders participate in market s, then Pesendorfer and Swinkels (1997)'s analysis implies that the auction price in market s converges to 1 and 0 in state $V=1$ and $V=0$, respectively, as the number of bidders n and the number of object $n\kappa_s$ grow arbitrarily large for any $g<1$. In other words, if $\kappa_r=0$, then information is aggregated because the auction price in market s converges to the object's value in each state.

As a second benchmark assume that $\kappa_r>0$ and suppose that all bidders receive perfectly informative signals ($g=0$). In this benchmark, there is a unique equilibrium for each n and information is again aggregated. In state $V=0$ all bidders bid zero in auction s because there is a positive reserve price in market r. Therefore, the price in state $V=0$ is equal to zero and c in markets s and r, respectively. In state $V=1$, the bidders randomize between the two auctions and bid one in the auction that they choose.[3] Since the bidders randomize, they are indifferent between the two markets in equilibrium. Moreover, the facts that all bidders bid one and $\kappa_r+\kappa_s<1$ together imply that the price in one of the two markets must converge to one. Since the bidders are indifferent between the two markets, the price in state $V=1$ must converge to one in both markets. Therefore, the auction price in market s converges to value and perfectly reveals the state.

In contrast to the two benchmarks that we studied above, we will now argue that price cannot converge to value in market s if there are sufficiently many uninformed

[3]Bidders bid their value in the auctions since the auctions are k_s+1 (or k_r+1) price auctions.

bidders.[4] For this argument we will assume that $1-g<\kappa_r$ and $c>1/2$. On the way to a contradiction, assume that price converges to value in auction s. No uninformed bidder and no bidder who knows that the state is $V=0$ would bid in market r in equilibrium because the price in this market is at least $c>1/2$ in both states. Consider a bidder who knows that the state is $V=1$. This bidder's payoff from participating in auction s converges to zero because the auction price converges to one in state $V=1$ by our initial assumption. The price in market r converges to c in both states because $1-g<\kappa_r$ and because only the informed bid in market r.[5] Therefore, any informed bidder will opt for market r in state $V=1$ for sufficiently large n. However, if no bidder other than the uninformed submit nontrivial bids that exceed zero in market s, then all the uninformed would bid $1/2$, i.e., their valuation for the object. Thus, the price cannot converge to one in state $V=1$, contradicting our initial assumption.

This example highlights the main tension between type dependent market selection and information aggregation. In order for information to be aggregated in market s, informed bidders must choose this market in both states. However, if information is aggregated, then no informed bidder would choose market s in state $V=1$ because they can obtain an object for a price equal to c in market r. In section 4, we characterize all equilibria for this example for any $c>0$, we show that the auction price perfectly reveals the state if and only if $\kappa_r<1-g$, and we describe the equilibrium mechanisms that lead to prices that do not perfectly reveal the state.

1.2. Relation to the Literature. We make two main contributions to the literature on information aggregation in multi-object common-value auctions. (1) We are the first to study bidding behavior in a multi-object common-value auction where bidders have outside options and the distribution of types is endogenously determined. (2) In this context, we highlight a new mechanism, based on self-selection, that can lead to the failure of information aggregation.

The model that we study is closest to the one studied by Pesendorfer and Swinkels (1997).[6] Their paper argued that prices converge to the true value of a common-value object in all symmetric equilibria if and only if both the number of objects and the num-

[4]A third benchmark that comes to mind is one where the reserve price c in market r is also equal to zero. In this case, price converges to value in both markets along every equilibrium sequence.

[5]In the auction that we study, if there are fewer bidders than objects, then the price is equal to the reserve price.

[6]There is extensive work on information aggregation by prices in various contexts. For example, see Wilson (1977) for common-value, uniform-price auctions with one object for sale; Pesendorfer and Swinkels (2000) for mixed private, common-value auctions; Reny and Perry (2006) and Cripps and Swinkels (2006) for large double auctions; Vives (2011) and Rostek and Weretka (2012) for markets for divisible objects; and Wolinsky (1990), Golosov et al. (2014), Ostrovsky (2012), Lauermann and Wolinsky (2015), and Lambert et al. (2018) for search markets and markets with dynamic trading.

ber of bidders who are not allocated an object grow without bound (double-largeness). In contrast, we show that information aggregation can fail if bidders have access to an outside option even when the double-largeness condition is satisfied.

Our paper is also related to recent work on single-unit common-value auctions by Lauermann and Wolinsky (2017) and Murto and Valimaki (2014). The novel feature of Lauermann and Wolinsky (2017)'s model is that the auctioneer knows the value of the object but must solicit bidders for the auction, and soliciting bidders is costly. Therefore, the number of bidders in the auction is endogenously determined by the auctioneer. Our paper differs from Lauermann and Wolinsky (2017) because: (1) We study a multi-unit multi-market auction, while they study a single-object single-market auction, and Pesendorfer and Swinkels (1997)'s analysis implies that the information aggregation properties of a multi-unit auctions differ substantially from the information aggregation properties of an auction with a single object. (2) In our model the distribution of types is determined by the participation decision of the bidders, while in their paper the auctioneer's solicitation strategy determines the number of bidders. This implies that in our model participation decisions are type dependent, while in theirs they are type independent but state dependent. In Murto and Valimaki (2014), potential bidders must pay a cost to participate in the auction. This creates type-dependent participation, as in our model. However, in contrast to this paper, they study a single-object, single-market auction and their emphasis is on characterizing equilibria rather than information aggregation.

Lauermann and Wolinsky (2017) and Atakan and Ekmekci (2014) also present models where information aggregation can fail in a large common-value auction. In both of these papers, information aggregation fails because there is an atom in the bid distribution (i.e., many types submit the same pooling bid) and the auction price is equal to this atom (pooling bid) with positive probability in both states of the world. In this paper, although the bid distribution may feature atoms, the failure of information aggregation is not caused by these atoms if the reserve price is small. In fact, we show that information aggregation fails because in market r there are more objects than bidders with positive probability in both states and because the same set of types determine the price in both states due to the pattern of self-selection in to market s. In fact, in the illustrative example we show that the limit-price distribution in market s is continuous, atomless, and increasing over the unit interval in both states.

2. Preliminaries

We study an auction where n bidders choose between three mutually exclusive alternatives: 1) A bidder can bid in market s; 2) She can bid in market r; or 3) She can choose neither and receive a payoff equal to zero. A bidder does not observe anything beyond her private signal when making this choice.

Market s is a common-value, sealed-bid, uniform-price auction for $\lceil \kappa_s n \rceil = k_s$ identical objects where $\kappa_s \in (0,1)$ is the object-to-bidder ratio.[7] There are $\lceil n\kappa_r \rceil = k_r$ additional objects on auction in market r and the auction format in market r is identical to market s except for a reserve price $c \in (0,1)$. The price in market s is equal to the k_s+1st highest bid in market s (the highest losing bid) if there are more bidders than objects and equal to zero, otherwise. The price in market r is equal to the maximum of c and the highest losing bid in market r if there are more bidders than objects and equal to c, otherwise. Ties are broken uniformly and randomly.

Each bidder has unit demand and puts value V on a single object, and value 0 on any further objects. The k_m highest bidders in auction $m \in \{r,s\}$ are allocated objects. Thus, a bidder who is allocated an object at price P enjoys utility $V-P$ while a bidder who fails to win an object receives a payoff equal to zero.

The unknown value $V \in \{0,1\}$ (or the state of the world) is common across players and drawn according to a common prior $\pi = 1/2$.[8] Before selecting a market, each bidder receives a signal $\theta \in [0,1]$ according to a continuous, increasing cumulative distribution function $F(\theta|v)$ that admits a density function $f(\theta|v)$, $v=0,1$.[9] Conditional on V, the signals are identically and independently distributed. Given that there are two states of the world, the signals satisfy the monotone likelihood ratio property **(MLRP)** possibly after a reordering. In other words, the likelihood ratio $l(\theta) := f(\theta|1)/f(\theta|0)$, is a nondecreasing function of θ. Throughout the paper, we further assume that (1) **there are no uninformative signals**, that is, $F(\{\theta : l(\theta)=1\})=0$; and (2) **signals contain bounded information**, i.e., there is a constant $\eta>0$ such that $\eta < l(\theta) < \frac{1}{\eta}$ for all $\theta \in [0,1]$. The first assumption states that the mass of signals that contain no information is equal to zero. This is a strengthening of MLRP, but it is weaker than assuming strict MLRP. The second assumption is a technical condition that is also maintained by Pesendorfer and Swinkels (1997). These assumptions significantly simplify the statements and proofs of our results. However, neither of these two assumptions is needed to show that information aggregation fails under the other assumptions outlined in the paper. In fact, in the illustrative example neither assumption is satisfied but all our of results nevertheless hold.

2.1. Strategies and Equilibrium. We represent bidder behavior by a symmetric distributional strategy H, which is a measure over $[0,1] \times \{s,r,neither\} \times [0,\infty)$. For a given symmetric strategy H, we define the **measure of types in auction** s as

[7]The smallest integer not less than x is denoted by $\lceil x \rceil$.

[8]We focus on a uniform prior for expositional simplicity only and none of our results depend on this assumption.

[9]For any half-open interval $(\theta',\theta'']$, we use $F((\theta',\theta'']|v) := F(\theta''|v) - F(\theta'|v)$.

$F_s^H(\theta):=H([0,\theta]\times\{s\}\times[0,\infty))$, we define the **selection function** $a^H:[0,1]\to[0,1]$ as the function such that $F_s^H(\theta)=\int_0^\theta a^H(\theta)dF(\theta)$.[10] Intuitively, $a^H(\theta)$ is the probability that type θ bids in auction s. Also, $F_s^H(\theta|v):=\int_0^\theta a^H(\theta)dF(\theta|v)$ is the measure of types that bid in market s conditional on $V=v$ and $\bar{F}_s^H(\theta|v):=F_s^H(1|v)-F_s^H(\theta|v)$. We focus on the symmetric Nash equilibria of the game Γ and we ignore, without loss of generality, the option of choosing "neither" because this option is never chosen by a positive measure of types in any symmetric equilibrium.[11]

The notation $\Pr^H$ represents the joint probability distribution over states of the world, signal and bid distributions, allocations, market choices, and prices, where this distribution is induced by the symmetric strategy H. We denote the payoff to type θ from bidding b in auction s if players are using strategy H by $u^H(s,b|\theta)$, and type θ's payoff under strategy H by $u^H(\theta)$. The kth highest type that bids in auction s is denoted by $Y_s^n(k)$, and we set $Y_s^n(k)$ equal to zero if there are fewer than k bidders in the auction.

The following lemma, which follows from Pesendorfer and Swinkels (1997, Lemmata 3-7), allows us to work exclusively with a pure and nondecreasing bidding strategy, i.e., a function $b:[0,1]\to[0,\infty)$ such that $H(\{\theta,s,b(\theta)\}_{\theta\in[0,1]})=F_s^H(1)$. Moreover, if the bidding function is increasing over an interval of types, then any type θ in this interval bids her value conditional on being the pivotal bidder in the auction.

Lemma 2.1. *Any equilibrium H can be represented by a nondecreasing bidding function b^H. Moreover, if $b^H(\theta)$ is increasing over an interval (θ',θ''), then*

$$b^H(\theta) \quad = \quad \mathbb{E}[V|Y_s^{n-1}(k_s)=\theta,\theta] \tag{2.1}$$

for almost every $\theta\in(\theta',\theta'')$.

Below we define a certain type $\theta_s^H(v)$ for each state v such that the expected number of bids above this type's bid in state v is exactly equal to the number of goods in market s. We refer to $\theta_s^H(v)$ as the **pivotal type in state** v because the types that determine the auction price are concentrated around $\theta_s^H(v)$ in a large market by the law of large numbers (LLN).

Definition 2.1 (Pivotal types)**.** For any symmetric strategy H, the **pivotal type** in state v is $\theta_s^H(v):=\max\{\theta:\bar{F}_s^H(\theta|v)=\kappa_s\}$, and $\theta_s^H(v):=0$ if the set is empty.[12]

[10]The unique function a^H is the Radon-Nikodym derivative of F_s^H with respect to F.

[11]If a positive mass of types were to choose "neither" in a symmetric equilibrium, then any bidder who submits a bid equal to zero in auction s would win an object with strictly positive probability in state $V=1$. Thus, all types who choose "neither" and receive a payoff equal to zero would rather bid zero in the auction and receive a strictly positive expected payoff.

[12]The equation $\bar{F}_s^H(\theta|v)=\kappa_s$ can have multiple solutions if F_s^H is flat over a range of θ. However,

For any sequence of strategies $\{H^n\}$, we will denote each $\theta_s^{H^n}(v)$ simply by $\theta_s^n(v)$, and we let $\theta_s(v) = \lim_n \theta_s^n(v)$ and $F_s(\theta|v) = \lim F_s^n(\theta|v)$ whenever such limits exist.[13]

2.2. Definition of Information Aggregation. We study a sequence of distributional strategies $\mathbf{H} = \{H^n\}_{n=1}^{\infty}$ for a sequence of auctions Γ^n where the n^{th} auction has n bidders. We assume that the parameters of the auctions are constant along the sequence and satisfy all the assumptions that we make.

Suppose that the number of bidders n is large. In this case, the LLN implies that observing the signals of all n bidders conveys precise information about the state of the world. A strategy H^n determines an auction price P^n given any realization of signals. We say that information is aggregated in the auction if this price also conveys precise information about the state of the world. Specifically, (i) if the likelihood ratio $l(P^n = p) := \frac{\Pr(V=1|P^n=p)}{\Pr(V=0|P^n=p)}$ is close to zero (i.e., if it is arbitrarily more probable that we observe such a price p when $V=0$), then an outsider who observes price p learns that the state is $V=0$. Alternatively, (ii) if the likelihood ratio $l(P^n=p)$ is arbitrarily large, then an outsider who observes price p learns that the state is $V=1$. If the probability that we observe a price that satisfies either (i) or (ii) is arbitrarily close to one, then we say that the equilibrium sequence aggregates information. Our formal definition of information aggregation is given below:

Definition 2.2. (Kremer (2002) and Atakan and Ekmekci (2014)) A sequence of strategies $\mathbf{H}$ aggregates information if the random variables $l(P^n=p)$ and $1/l(P^n=p)$ converge in probability to zero in state 0 and state 1, respectively.

We now derive conditions that are necessary and sufficient for information aggregation. Information aggregation fails if the supports of the limit price distributions are the same in the two states. The following definition captures such failures using the mass that separates the pivotal types.

Definition 2.3. The pivotal types are **distinct** along a sequence $\mathbf{H}$ if $\lim_n \sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1|) \to \infty$ and the pivotal types are **arbitrarily close** along a sequence $\mathbf{H}$ if $\liminf_n \sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| < \infty$.[14]

Distinct pivotal types is a necessary condition for information aggregation. To see why, recall that the random variable $Y_s^n(k_s+1)$ denotes the k_s+1st highest type that bids

the function $\bar{F}_s^H(\theta|v)$ is continuous because it is absolutely continuous with respect to $\bar{F}(\theta|v)$. Hence, the set $\{\theta : \bar{F}_s^H(\theta|v) = \kappa_s\} \subset [0,1]$ is compact and has a unique maximal element if it is nonempty.

[13] Such limits always exist along a subsequence.

[14] In Lemma A.5 in the Appendix we show that $\lim \sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| < \infty$ if and only if $\lim \sqrt{n}|F_s^n(\theta_s^n(1)|0) - F_s^n(\theta_s^n(0)|0)| < \infty$ utilizing our assumption that there are no arbitrarily informative signals.

in the auction. The auction clears at the bid of this type because bidding is monotone (Lemma 2.1). For large n, the distribution of $Y^n_s(k_s+1)$ in state $V=v$ puts most of the mass within finitely many standard deviations of the pivotal type in state $V=v$ and the standard deviation is on the order of $\sqrt{1/n}$. If the pivotal types are arbitrarily close, i.e., if the pivotal types are separated by finitely many standard deviations, then the same set of types determine the price and the supports of the limit price distributions are the same in the two states. Therefore, information cannot be aggregated.

Information aggregation also fails if the limit price distribution features an atom that occurs with positive probability in both states. We term such a failure pooling by pivotal types and formally define it below.

Definition 2.4. There is **pooling by pivotal types** along a sequence $\mathbf{H}$ if there is a subsequence of pooling bids $\{b_p^{n_k}\}$ such that $\lim_k \Pr(P^{n_k}=b_p^{n_k}|V=v)>0$ for $v=0,1$. Otherwise, there is **no pooling by pivotal types**.

No pooling by pivotal types is also a necessary condition for information aggregation because if it does not hold, then the limit price distribution features an atom that occurs with positive probability in both states. In the following lemma, we further show that these two necessary conditions are also sufficient for information aggregation.

Lemma 2.2. *An equilibrium sequence aggregates information if and only if the pivotal types are distinct and there is no pooling by pivotal types.*

A sketch of the argument for sufficiency is as follows: Pick any type θ that is within finitely many standard deviations of the pivotal type in state $V=1$ and note that the auction can clear only at the bids of such types in state $V=1$. Distinctness of the pivotal types implies that type θ is infinitely many standard deviations away from the pivotal type in state $V=0$. Therefore, if type θ does not bid in an atom, then an outside observer, who observes a price equal to type θ's bid, is arbitrarily certain that the state is $V=1$. On the other hand, suppose that θ bids in an atom, i.e., suppose that the price is equal to θ's bid with positive probability in state $V=1$. In this case, the probability that the price is equal to θ's bid in state $V=0$ is equal to zero because there is no pooling by pivotal types. Once again, an outside observer, who observes a price equal to θ's bid, is arbitrarily certain that the state is $V=1$.

3. Information Aggregation

This section's main theorem shows that information is not aggregated in market s along any equilibrium sequence if the object-to-bidder ratio in market s exceeds a certain cutoff $\bar{\kappa}$ (described further below). Conversely, if the object-to-bidder ratio in market s is less than $\bar{\kappa}$, then information is aggregated in market s along every equilibrium sequence.

In order to state our main theorem, we first define the cutoff $\bar{\kappa}$. Let $\theta_r^F(1)$ denote the pivotal type in market r in state $V=1$ if all types were to bid in auction r, that is, $\theta_r^F(1)$ is the unique type that satisfies the equality $1-F(\theta_r^F(1)|1)=\kappa_r$. For a given type $\theta'<1$, let $\theta^*(\theta')$ denote the unique type $\theta<\theta'$ such that $F([\theta,\theta']|0)=F([\theta,\theta']|1)$, and let $\theta^*(\theta')=\theta'$ if there is no such $\theta<\theta'$.[15] For some intuition, suppose that types $\theta>\theta'$ opt for market r, while types $\theta\leq\theta'$ bid in auction s. In this case, $\theta^*(\theta')$ is defined as the type such that the expected number of bidders who bid in auction s with signals that exceed $\theta^*(\theta')$ is the same in both states. The implicit function theorem and MLRP together imply that $\theta^*(\theta')$ is a decreasing function of θ'.[16]

Definition 3.1. Let $\theta_{en} := \max\{\theta_r^F(1), \inf\{\theta : \Pr(V=1|\theta) > c\}\}$ and $\theta_{en} := 1$ if the set over which the infimum is taken is empty. Define $\bar{\kappa} := F([\theta^*(\theta_{en}),\theta_{en}]|0) = F([\theta^*(\theta_{en}),\theta_{en}]|1)$.

To better understand the definition of $\bar{\kappa}$, suppose that all types greater than θ_{en} select market r while all types smaller than θ_{en} bid in market s. The cutoff $\bar{\kappa}$ is defined so that if the object-to-bidder ratio in market s is equal to $\bar{\kappa}$, then the pivotal type in market s is equal to $\theta^*(\theta_{en})$ in both of the states. Turning next to the definition of θ_{en}, further suppose that any type that chooses market r bids according to an increasing bidding function. Type θ_{en} is defined as the smallest type that can make positive profits in an arbitrarily large market r. To see why the definition captures this property, note that θ_{en} must be at least as large as $\theta_r^F(1)$ because only those types greater than $\theta_r^F(1)$ can actually win an object in the auction in state $V=1$. Furthermore, any type $\theta>\theta_r^F(1)$ can make a profit in market r only if $\Pr(V=1|\theta)>c$ because any such type will win an object with probability one in both states and will pay a price which is at least c. Also, see Figure 3.1 for a graphical depiction of $\bar{\kappa}$.

The main implication of Definition 3.1 is as follows: if the object-to-bidder ratio in market s exceeds $\bar{\kappa}$, then the pivotal type in state 0 exceeds the pivotal type in state 1 whenever all bidders who value the outside option opt for market r. Such an ordering of pivotal types is ruled out by MLRP if all types were to bid in market s. However, if types that exceed $\theta_{en}<1$ choose market r, then the measure of types that bid in market s is smaller in state 1 than in state 0 as a consequence of MLRP. This implies that $\bar{\kappa}$ is less than one. Therefore, there is an open interval $(\bar{\kappa},1)$ such that whenever

[15] More precisely, if $l(\theta')>1$, then there is a unique type $\theta<\theta'$ such that $F([\theta,\theta']|0)=F([\theta,\theta']|1)$. Otherwise, there is no such type and $\theta^*(\theta')=\theta'$.

[16] If $l(\theta')\leq 1$, then $\theta^*(\theta')=\theta'$ and the function is decreasing. Otherwise, $F([\theta^*(\theta'),\theta']|0) = F([\theta^*(\theta'),\theta']|1)$ and the implicit function theorem implies that $d\theta^*/d\theta' = f(\theta'|0)(l(\theta')-1)/f(\theta^*|0)(l(\theta^*)-1)$. The fact that $F([\theta^*(\theta'),\theta']|0)=F([\theta^*(\theta'),\theta']|1)$ and MLRP together imply that $l(\theta')<1$. Moreover, if $\theta^*(\theta')<\theta'$, then MLRP implies that $l(\theta^*)>1$. Therefore, $d\theta^*/d\theta'<0$.

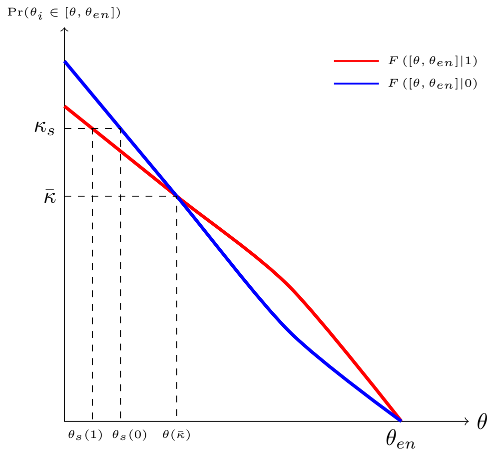


Figure 3.1: The function $F([\theta,\theta_{en}]|v)$ depicts the fraction of types above θ that bid in auction s in state v given that all types $\theta > \theta_{en}$ take the outside option. The cutoff $\bar{\kappa}$ is defined as the value of $F([\theta,\theta_{en}]|v)$ at the point $\theta < \theta_{en}$ where $F([\theta,\theta_{en}]|1)$ and $F([\theta,\theta_{en}]|0)$ cross. If $\kappa_s > \bar{\kappa}$, then the pivotal type in state 0 exceeds the pivotal type in state 1.

the object-to-bidder ratio is in this interval, the order of the pivotal types is reversed. The converse is also true, that is, if the object-to-bidder ratio in auction s is less than $\bar{\kappa}$, then the pivotal type in state 1 exceeds the pivotal type in state 0 even if all types that value the outside option opt for market r.

Our main theorem is stated below:

Theorem 3.1. *If the object-to-bidder ratio in market s exceeds $\bar{\kappa}$, then there is no equilibrium sequence that aggregates information in either market. If the object-to-bidder ratio in market s is less than $\bar{\kappa}$, then information is aggregated in market s along any equilibrium sequence.*

The argument for our main theorem shows that information cannot be aggregated along any equilibrium sequence in market s if the order of the pivotal types in this market is reversed whenever all bidders who value the outside option select the outside option, i.e., if $\kappa_s > \bar{\kappa}$. In other words, self-selection is detrimental to information aggregation when competition for objects in market s is sufficiently low, or equivalently, when the object-to-bidder ratio in market s is below the threshold $\bar{\kappa}$. Conversely, information is aggregated in market s along any equilibrium sequence if the order of the pivotal types is preserved even when all the bidders who value the outside option select the outside option, i.e., if competition for objects in market s is sufficiently high.

Before providing some intuition for Theorem 3.1, we describe an intermediate result that we utilize: if information is aggregated in market s, then the price in market s converges to zero in state $V = 0$ and one in state $V = 1$, i.e., price converges to value.

In order to provide an argument for this intermediate result, we first note that Lemma 2.2 implies that there is a bid $b^* > 0$ that separates the support of the limit-price distribution in state 0 from the support of the limit-price distribution in state 1 if information is aggregated in market s.

The first step of the argument that establishes the intermediate result stated above shows that the limit-price distribution's support lies below b^* in state 0 and above b^* in state $V=1$: Suppose, on the way to a contradiction, that the limit-price distribution's support lies above b^* in state $V=0$ and below b^* in state $V=1$. Then, any bidder can ensure that she wins an object only in state $V=1$ with probability one by submitting a bid equal to b^*. Therefore, any bidder that submits a bid greater than b^* can improve her payoff by instead submitting a bid equal to b^*. So, the limit-price distribution's support cannot lie above b^* in state 0. The second step argues that bids less than b^* must all converge to zero, and therefore the price in state 0 must converge to zero: Any bid less than b^* never wins in state $V=1$ and therefore any such bid, and in particular, the bid of the pivotal type in state $V=0$ must converge to zero. The final step concludes that the price in state $V=1$ must converge to one. If the expected price in state 1 is strictly less than one, then the pivotal type in state 0 could improve her payoff by bidding one instead of following her equilibrium strategy. If she follows her equilibrium strategy, she never wins an object in state $V=1$ and receives a payoff equal to zero, while under the deviation she wins an object at a price equal to zero in state $V=0$ and at a price which is strictly less than one in state $V=1$ with positive probability.

Intuition for why information is not aggregated in market s if $\kappa_s > \bar{\kappa}$. On the way to a contradiction, assume that price converges to value in market s and therefore the payoff of any type that bids in market s is equal to zero. If this is so, then all types that exceed θ_{en} would opt for market r. To see this, observe that if any type $\theta > \theta_{en}$ did not choose market r, then less optimistic types would not choose market r either. Moreover, at the limit, types that exceed θ_{en} face a choice between market s, where their payoff is equal to zero, and market r, where their payoff is positive (in fact, their payoff is equal to $-c$ if $V=0$ and $1-c$ if $V=1$). However, if all types that exceed θ_{en} opt for market r and if $\kappa_s > \bar{\kappa}$, then we find $\theta_s(0) > \theta_s(1)$ (see figure 3.1). If information is aggregated in market s, then $\lim_{n\to\infty} b^n(\theta_s^n(1)) = 1$ and $\lim_{n\to\infty} b^n(\theta_s^n(0)) = 0$ because price converges to value. However, this leads to a contradiction that proves the result. The findings that $\lim_{n\to\infty} b^n(\theta_s^n(1)) = 1$, $\lim_{n\to\infty} b^n(\theta_s^n(0)) = 0$, and $\theta_s(0) > \theta_s(1)$ together contradict that the bidding function is nondecreasing in θ for all n. Intuitively, more pessimistic types opt for the auction and there are more of such types in state $V=0$. Therefore, the auction clears at the bid of a more pessimistic type in state $V=1$ than in state $V=0$ and this is incompatible with price converging to value.

Intuition for why information is not aggregated in market r. In market r information aggregation fails for any κ_s in contrast to market s. A similar argument to the one given for market s implies that the price in market r converges to one in state $V=1$ if information is aggregated. However, if price in market r converges to one in state $V=1$, then the payoff from bidding in market r is negative for all types and therefore no type would choose this market. But if no type chooses this market, then the price is equal to c in both states and information is not aggregated in market r.

Recall that information is aggregated in an auction if and only if the pivotal types are distinct and they submit distinct bids (no pooling by pivotal types) by Lemma 2.2. Our argument above showed that information aggregation fails whenever $\kappa_s > \bar{\kappa}$. Therefore, if $\kappa_s > \bar{\kappa}$, then information aggregation must fail in market s either because the pivotal types are arbitrarily close or because the pivotal types bid in an atom. In section 4, we use versions of the illustrative example to construct equilibria where the pivotal types are arbitrarily close and an equilibrium where the pivotal types bid in an atom.

Intuition for why information is aggregated in market s if $\kappa_s < \bar{\kappa}$. The definition of $\bar{\kappa}$ implies that $\theta_s(1) > \theta_s(0)$ whenever $\kappa_s < \bar{\kappa}$, i.e., the pivotal types are distinct. Below, we argue that there can be no pooling by pivotal types either whenever $\theta_s(1) > \theta_s(0)$. But then Lemma 2.2 implies that information is aggregated.

To sustain a pool, the highest type that submits the pooling bid (denoted by $\bar{\theta}_p$) must prefer the pooling bid to a slightly higher bid that wins an object with probability one whenever the price is equal to the pooling bid. Also, the lowest type that submits the pooling bid (denoted by $\underline{\theta}_p$) must prefer the pooling bid to a slightly lower bid that avoids winning an object whenever the price is equal to the pooling bid. In other words, pooling must be incentive compatible for type $\bar{\theta}_p$ and individually rational for type $\underline{\theta}_p$. In the terminology of Lauermann and Wolinsky (2017) (or Pesendorfer and Swinkels (1997)), we say that there is **winner's blessing** at pooling if the probability of winning at the pooling bid is higher when $V=1$ than when $V=0$, in other words, if a bidder wins more frequently at pooling when the object's value is high. Similarly, there is **loser's blessing** at pooling if a bidder loses more frequently at pooling when the object's value is low. Put another way, if there is loser's and winner's blessing at pooling, then losing is a signal in favor of $V=0$ and winning a signal in favor of $V=1$. The strengths of these two signals determine whether a pooling bid is incentive compatible and individually rational. In particular, the loser's blessing's strength determines the lowest pooling bid that is incentive compatible for type $\bar{\theta}_p$ while the winner's blessing's strength determines the highest pooling bid that is individually rational for type $\underline{\theta}_p$.

Our key result that establishes that pooling by pivotal types is not possible shows that if $\theta_s(1) > \theta_s(0)$, then there are bounds on the strength of the loser's and winner's blessing at the pooling bid. These bounds preclude a pooling bid that is both individually rational for type $\underline{\theta}_p$ and incentive compatible for type θ_p thus establishing that pooling by pivotal types is incompatible with equilibrium.

The cutoff object-to-bidder ratio $\bar{\kappa}$, which determines whether information is aggregated, is a function of the signal distribution, the reserve price c, and κ_r. We end this section by providing some comparative statics for the cutoff object-to-bidder ratio $\bar{\kappa}$ in the following remark. We also comment on two parameter values that are not explicitly covered by Theorem 3.1: the case where all bidders are perfectly informed and the case where $c=0$.

Remark 3.1. The ratio $\bar{\kappa}$ is non-decreasing in c and non-increasing in κ_r. This is because the type θ_{en} is non-decreasing in c and non-increasing in κ_r. Consequently, $\theta^*(\theta_{en})$ is non-increasing in c and non-decreasing in κ_r. If no type finds it profitable to purchase an object at a price equal to c, i.e., if $c > \Pr(V=1|\theta)$ for all θ, then $\theta^*(\theta_{en}) = 1$ and $\bar{\kappa} = 1$. If all types are perfectly informed or if $c=0$, then information is aggregated in both markets if $\kappa_s + \kappa_r < 1$. As one would expect, information is not aggregated in either market if $\kappa_s + \kappa_r > 1$ in both of these cases.[17]

4. Equilibrium Causes for Non-informative Prices

Theorem 3.1 showed that information is not aggregated whenever κ_s is sufficiently large. However, the argument for the theorem does not fully convey the underlying mechanism that delivers non-revealing prices. In this section, we identify the features of equilibrium sequences that fail to aggregate information using the illustrative example. At the end of the section, we show that the features that we highlight in this discussion are shared by equilibrium sequences under general signal structures.

There are three different equilibrium outcomes in which information is not aggregated in a market: 1) *Lack of competition*: In this outcome, the number of objects in the auction exceeds the number of bidders that submit a positive bid in the auction with positive probability. This implies that the pivotal types are arbitrarily close and information aggregation fails because the price is equal to zero (or the reserve price) with positive probability in both states. 2) *Arbitrarily close pivotal types with sufficient competition*: In this outcome, there are more bidders than objects in both states with probability one but the pivotal types in the auction are arbitrarily close. Information aggregation fails because the same set of types determine the price in both states and

[17] If all types are perfectly informed, then a straightforward computation yields $\bar{\kappa} = 1 - \kappa_r$. Also, see Remark 4.1 for more on this case. The proof of our claim relating to the case where $c=0$ is available in the working paper version of this paper (Atakan and Ekmekci (2020).

thus the limit price distributions in the two states have the same support. 3) *Pooling by pivotal types*: In this outcome, the pivotal types submit the same bid, and the auction price is equal to this pooling bid with positive probability in both states. Lemma 2.2 implies that this classification of asymptotic outcomes is exhaustive.

4.1. Illustrative Example. Throughout this subsection we assume that $\kappa_s+\kappa_r<1$ and we parameterize the illustrative example's signal structure using the density function below:

$$f(\theta|1)=\begin{cases}0 & \text{for } \theta\in[0,1/3)\\ 3g & \text{for } \theta\in[1/3,2/3]\\ 3(1-g) & \text{for } \theta\in(2/3,1]\end{cases} \quad f(\theta|0)=\begin{cases}3(1-g\frac{(1-\pi)}{\pi}) & \text{for } \theta\in[0,1/3)\\ 3g\frac{(1-\pi)}{\pi} & \text{for } \theta\in[1/3,2/3]\\ 0 & \text{for } \theta\in(2/3,1]\end{cases}$$

where $\pi\in[0,1]$. Types $\theta\in\mathcal{E}(1):=(2/3,1]$ know for certain that the state is $V=1$, types $\theta\in\mathcal{E}(0):=[0,1/3)$ know for certain that the state is $V=0$, and $\Pr(V=1|\theta)=\pi$ for types $\theta\in\mathcal{E}(1/2):=[1/3,2/3]$. If $\pi=1/2$, then types in $\mathcal{E}(1/2)$ are uninformed and the mass of uninformed types is equal to g as in the illustrative example presented in the introduction.

In the next proposition we characterize all equilibria for the illustrative example. We show that information aggregation fails in both markets if $1-g<\kappa_r$. In market s information aggregation fails because the pivotal types are arbitrarily close even though there is sufficient competition; while in market r information aggregation fails because there is lack of competition. If $1-g>\kappa_r$, then information is aggregated in both markets.

In order to state the result, we define a type's expected value conditioning on the event of being pivotal in market m:

$$b^n_m(\theta):=\frac{\left(\frac{1-\bar{F}^n_m(\theta|0)-F^n_m(\mathcal{E}(1)|1)}{1-\bar{F}^n_m(\theta|0)}\right)^{n-k_s-1}\left(\frac{\bar{F}^n_m(\theta|0)+F^n_m(\mathcal{E}(1)|1)}{\bar{F}^n_m(\theta|0)}\right)^{k_s-1}}{1+\left(\frac{1-\bar{F}^n_m(\theta|0)-F^n_m(\mathcal{E}(1)|1)}{1-\bar{F}^n_m(\theta|0)}\right)^{n-k_s-1}\left(\frac{\bar{F}^n_m(\theta|0)+F^n_m(\mathcal{E}(1)|1)}{\bar{F}^n_m(\theta|0)}\right)^{k_s-1}} \tag{4.1}$$

for each $\theta\in\mathcal{E}(1/2)$, $b^n_m(\theta)=0$ for each $\theta\in\mathcal{E}(0)$, and $b^n_m(\theta)=1$ for each $\theta\in\mathcal{E}(1)$. In this calculation, we have used the fact that $\bar{F}^n_m(\theta|1)=\bar{F}^n_m(\theta|0)+F^n_m(\mathcal{E}(1)|1)$ for each $\theta\in\mathcal{E}(1/2)$. Note that the function $b^n_m(\theta)$ is increasing at each $\theta\in\mathcal{E}(1/2)$ if $F^n_m(\mathcal{E}(1)|1)>0$ and $b^n_m(\theta)=1/2$ for each $\theta\in\mathcal{E}(1/2)$ if $F^n_m(\mathcal{E}(1)|1)=0$. The proposition below summarizes the equilibria of the illustrative example.

Proposition 4.1. *Assume $\kappa_s+\kappa_r<1$ and $\pi=1/2$. An equilibrium exists for each n. In any equilibrium, a positive measure of types $\theta\in\mathcal{E}(1)$ choose market s for all sufficiently large n; any type θ that chooses market m submits a bid equal to $b^n_m(\theta)$; and $b^n_r(\theta)\to 1$ for all θ that select market r. If $1-g>\kappa_r$, then information is aggregated in both markets. If $1-g<\kappa_r$, then any equilibrium sequence converges to a unique* *outcome that satisfies the following properties:*

i. There is lack of competition in market r: $\lim F_r^n(1|1) \leq \kappa_r$ *and* $\lim F_r^n(1|0) < \kappa_r$. *Market r's price converges to c with probability one if* $V=0$ *and converges to a random variable* P_r *that is equal to c with positive probability and equal to one with the remaining probability if* $V=1$.

ii. The pivotal types in market s are arbitrarily close and the measure of types $\theta \in \mathcal{E}(1)$ *that select market s converges to zero.*

iii. The price in market s converges in distribution to a random variable P_s. *If* $c \neq 1/2$, *then the distribution functions* $\Pr[P_s \leq p|V=1]$ *and* $\Pr[P_s \leq p|V=0]$ *are atomless and increasing on the unit interval. If* $c=1/2$, *then* $P_s=1/2$.

iv. If $c \geq 1/2$, *then only types* $\theta \in \mathcal{E}(1)$ *choose market r. In this case* $\mathbb{E}[P_r|V=1] = \mathbb{E}[P_r|V=0]=c$, $\mathbb{E}[P_s|V=1]=c$, *and* $\mathbb{E}[P_s|V=0]=1-c$.

v. If $c < 1/2$, *then the mass of types* $\theta \in \mathcal{E}(1/2)$ *that choose market r converges to* $\kappa_r + g - 1$. *In this case,* $\mathbb{E}[P_r|V = 1] = \mathbb{E}[P_s|V = 1] = 1 - c$, *and* $\mathbb{E}[P_r|V=0]=\mathbb{E}[P_s|V=0]=c$.

Remark 4.1. If $g=0$, i.e., if all the bidders are perfectly informed, then the unique equilibrium in the auction is again described by the proposition above and information is aggregated along the unique equilibrium sequence in both markets because $g=0$ implies that $1-g>\kappa_r$.

It is worthwhile to highlight properties of the equilibria in which information aggregation fails ($1-g<\kappa_r$): The mass of types $\theta \in \mathcal{E}(1)$ that choose market s is positive for all sufficiently large n but converges to zero. This implies that types $\theta \in \mathcal{E}(1)$ are indifferent between the two markets. Also, the mass of uninformed types that choose market s exceeds κ_s at the limit. Therefore, the probability that the number of bidders in market s exceeds the number of objects converges to one in both states. This rules out *lack of competition* in market s. Moreover, each uninformed type $\theta \in \mathcal{E}(1/2)$ that chooses market s submits a bid equal to $b_s^n(\theta)$ for each n and the price in market s is equal to the bid of an uninformed type with probability converging to one. The fact that $b_s^n(\theta)$ is increasing rules out pooling by pivotal types in market s thus *arbitrarily close pivotal types with sufficient competition* is the only possible asymptotic outcome in market s.

Turning attention to market r, we find that the mass of types that bid in market r is less than κ_r in state $V=0$. Therefore, the price converges to c in state $V=0$. If $c \geq 1/2$, then no uninformed type bids in market r, the mass of types that bid in market r is less than κ_r in both states, and the price converges to c in state $V=1$ also. If $c<1/2$, then in state $V=1$ the mass of types that bid in market r converges to κ_r and the price is equal to c with positive probability. Therefore, information aggregation fails in market r because of *lack of competition*.

The expected prices in the two markets depend on the reserve price c. If $c>1/2$, then only types $\theta\in\mathcal{E}(1)$ are indifferent between the two markets and all other types prefer market s. This implies that expected prices are equalized across the two markets in only state $V=1$. On the other hand if $c<1/2$, then both uninformed types and types $\theta\in\mathcal{E}(1)$ are indifferent between the two markets. This implies that the expected prices are equalized across the two markets in both states.

Intuition for the Proposition. Our characterization of the limit outcome is based on two properties shared by equilibrium sequences: (1) *All types θ that choose market $m\in\{s,r\}$ submit a bid equal to $b^n_m(\theta)$;* and (2) *These bidding functions' limits are uniquely determined by the mass of types $\theta\in\mathcal{E}(1)$ that bid in market s.* These two properties imply that once we know how types select across markets, we can compute the limit bid distributions and therefore the price distributions in both markets. We now elaborate on why these properties hold and how they deliver the limit outcome described by the proposition.

Property 1. Types θ that choose market m submit a bid equal to $b^n_m(\theta)$. The informed types know the state and therefore bid their value. We now argue that all the uninformed bid according to $b^n_m(\theta)$ also. If a positive mass of types $\theta\in\mathcal{E}(1)$ choose market m, then the uninformed cannot bid in an atom because they are subject to the loser's curse at an atom. However, if the uninformed do not bid in an atom, then the bidding function is increasing and therefore given by Eq. 4.1 (see Lemma 2.1). Alternatively, if no type $\theta\in\mathcal{E}(1)$ chooses market m, then each $\theta\in\mathcal{E}(1/2)$ would submit a bid equal to $1/2$ in market m again showing that Eq. 4.1 is satisfied.

Property 2. The bidding function in market m converges to a limit which is uniquely determined by the mass of types $\theta\in\mathcal{E}(1)$ that bid in market s. Without loss of generality we focus on market s. Pick a sequence of types $\{\theta^n\}\subset\mathcal{E}(1/2)$ such that each type θ^n in the sequence is z standard deviations away from the pivotal type $\theta^n_s(1)$ in state $V=1$, i.e., $z=(F^n_s(\theta^n|1)-F^n_s(\theta^n_s(1)|1))/\sigma^n$ where $\sigma^n\approx\sqrt{(1-\kappa_s)\kappa_s/n}$. A simple calculation shows that the mass that separates the pivotal types in the two states is equal to the mass of informed types that bid in market s in state $V=1$ (i.e., $F^n_s(\theta^n_s(1)|1)-F^n_s(\theta^n_s(0)|1)=F^n_s(\mathcal{E}(1)|1)$). Consequently, type θ^n is $z+F^n_s(\mathcal{E}(1)|1)/\sigma^n$ standard deviations away from the pivotal type $\theta^n_s(0)$ in state $V=0$. The central limit theorem implies that

$Y_s^{n-1}(k_s)$ is asymptotically normal and centered around $\theta_s(v)$ in state $V=v$. Therefore,

$$b_s^n(\theta^n)=\frac{\frac{\Pr(Y_s^{n-1}(k_s)=\theta^n|V=1)}{\Pr\left(Y_s^{n-1}(k_s)=\theta^n|V=0\right)}}{1+\frac{\Pr(Y_s^{n-1}(k_s)=\theta^n|V=1)}{\Pr\left(Y_s^{n-1}(k_s)=\theta^n|V=0\right)}}\to b_s(z):=\frac{\frac{\phi(z)}{\phi(z+x)}}{1+\frac{\phi(z)}{\phi(z+x)}}=\frac{e^{xz+\frac{x^2}{2}}}{1+e^{xz+\frac{x^2}{2}}}.$$

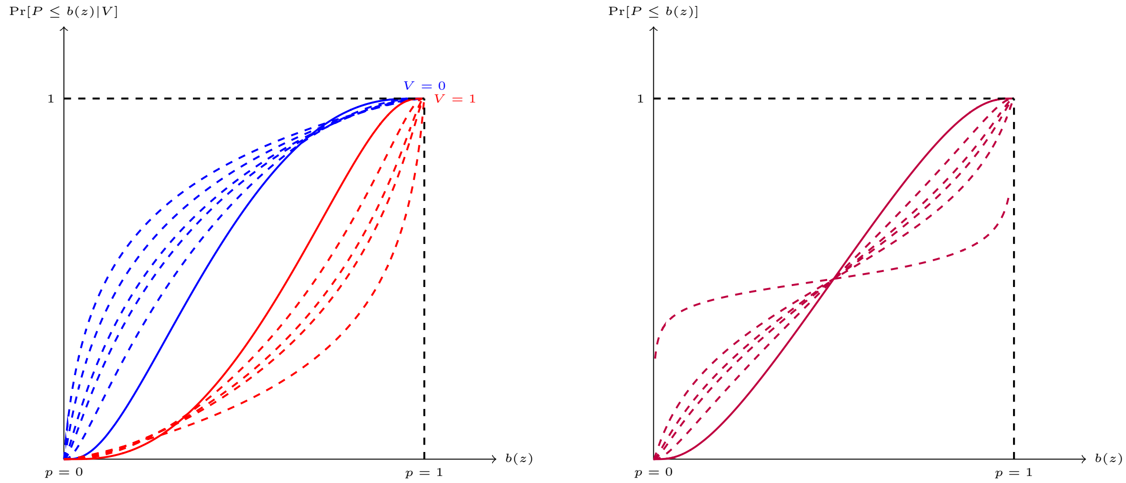
where ϕ is the standard normal density and $x:=\lim F_s^n(\mathcal{E}(1)|1)/\sigma^n$. The central limit theorem further implies that the auction price is less than or equal to the bid of type θ^n with probability $\Phi(z)$ and $\Phi(z+x)$ in states $V=1$ and $V=0$, respectively. Moreover, the price in state $V=1$ clears at the bid of a type that is finitely many standard deviations away from the pivotal type in state $V=1$ with probability converging to one.

We now sketch how an equilibrium sequence described by the proposition is sustained at the limit using these two properties.

First suppose $c>1/2$. It is straightforward to see that no uninformed type and no type $\theta\in\mathcal{E}(0)$ would choose market r because any such type's payoff in market r is negative. Moreover, the price in market r is equal to c in both states because only types $\theta\in\mathcal{E}(1)$ choose market r. We will further argue that $\lim F_s^n(\mathcal{E}(1)|1)/\sigma^n=x\in(0,\infty)$. In other words, the mass of types $\theta\in\mathcal{E}(1)$ that choose market s is positive for all sufficiently large n but converges to zero at the rate of $1/\sqrt{n}$. Therefore, the pivotal types in market s are arbitrarily close.

Suppose that $x=\infty$. In this case, the limit bid function $b_s(z)$ (introduced in Property 2) is equal to one for any finite z and therefore the price in market s converges to one in state $V=1$. However, then all types $\theta\in\mathcal{E}(1)$ would prefer market r for sufficiently large n because the price in market r converges to $c<1$. This contradicts $x=\infty$. If, on the other hand, $x=0$, then $b_s(z)=1/2$ for any finite z and therefore the price in market s converges to $1/2$ in state $V=1$. However, then all $\theta\in\mathcal{E}(1)$ would prefer to bid in market s because the price in market r is equal to $1/2<c$. This contradicts $x=0$. The fact that $x\in(0,\infty)$ implies that types $\theta\in\mathcal{E}(1)$ are indifferent between the two markets for all sufficiently large n and therefore the expected prices in the two markets are equal in state $V=1$. Moreover, the intermediate value theorem implies that there is a value of $x\in(0,\infty)$ such that the expected price in market s in state $V=1$ is equal to the expected price in market r, which converges to c. In the proof, we further show that this value of x is unique. Figure 4.1 depicts the limit price distribution in market s.

Now suppose $c<1/2$. Similar to the previous case, $\lim F_s^n(\mathcal{E}(1)|1)/\sigma^n=x\in(0,\infty)$. Therefore, the expected prices in the two markets must be equal in state $V=1$ at the limit. Additionally, the mass of uninformed types that select markets r and s converge to $\kappa_r+g-1\in(0,\kappa_r)$ and $1-\kappa_r>\kappa_s$, respectively. Therefore, the uninformed types must



(a) Limit price distributions in states $V=0$ (the curves on the left) and $V=1$ (the curves on the right).

(b) Unconditional limit price distribution.

Figure 4.1: The solid curves are the cumulative price distributions for $c=0.6$. If $c=0.6$, we numerically find that x is approximately equal to one, i.e., the pivotal types are separated by one standard deviation. The dotted lines give the price distributions as c ranges from 0.6 to 0.8 and therefore as x ranges approximately from 1 to 2. As c approaches one, the outside option's value approaches zero in both states; the price distribution in state $V=1$ converges to a point mass at $p=1$; and the price distribution in state $V=0$ converges to a point mass at $p=0$, i.e., price aggregates information.

also be indifferent between the two markets. This implies that the expected prices in both markets are equal in state $V=0$ also.

Note that the price in market r converges to c in state $V=0$ because the total mass of types that bid in market r in state $V=0$ is less than κ_r. Therefore, the expected price in market s also converges to c in state $V=0$. Moreover, the expected price in state $V=1$ converges to $1-c$ in both markets. To see this, note that there is an uninformed type θ that bids in market s and wins an object with probability converging to zero. This is because the mass of uninformed types that choose market s exceeds κ_s and bidding is increasing by Property 1. This type's payoff converges to zero and consequently all uninformed types' payoffs converge to zero. There is also an uninformed type $\hat{\theta}$ that wins an object from market s with probability converging to one in both states. In order to ensure that type $\hat{\theta}$'s payoff converges to zero, the expected price in state $V=1$ must converge to $1-c$ because the expected price converges to c in state $V=0$.

To complete the argument for market s, we show that there is a value of $x \in (0,\infty)$ such that the expected price in market s in state $V=0$, which converges to $\int_{-\infty}^{\infty} b(z)d\Phi(z+x)$, is equal to the expected price in market r, which converges to

c. We show that the expected price in market s in state $V=1$, which converges to $\int_{-\infty}^{\infty} b(z)d\Phi(z)$, is equal to $1-c$ for the same value of x .

Again turning attention to market r, we find that the bids of all types that bid in market r converge to one. This is a consequence of Property 2: the mass of types $\theta \in \mathcal{E}(1)$ that bid in market r converges to g since the mass of such types that bid in market s converges to zero. Property 2 implies that $b_r(z)=1$ for all z since $\sqrt{n}F_r^n(\mathcal{E}(1)|1)$ converges to infinity. In other words, the bids of all the uninformed types in market r converge to one. The price in state $V=1$ clears at a bid with probability converging to $(1-2c)/(1-c)$ and, with the remaining probability, clears at the reserve price because there are fewer bidders than there are objects. Hence, the price in state $V=1$ converges to a binary random variable that is equal to one and c with probabilities $(1-2c)/(1-c)$ and $c/(1-c)$, respectively.

We end this subsection by describing how information is aggregated if $F(\mathcal{E}(1)|1)= 1-g>\kappa_r$: First, suppose that the mass of types in $\mathcal{E}(1)$ that select market s converges to zero. Then the price in market r converges to one in state $V=1$ because $\lim F_r^n(\mathcal{E}(1)|1)=1-g>\kappa_r$. This implies that the price in market s also converges to one because otherwise all types in $\mathcal{E}(1)$ would select market s. Moreover, $P_r^n \to 1$ in state $V=1$ implies that no uninformed type selects market r for sufficiently large n and therefore the price in state $V=0$ converges to c, i.e., information is aggregated in market r. Furthermore, $P_s^n \to 1$ in state $V=1$ implies that $P_s^n \to 0$ in state $V=0$ because otherwise any uninformed type bidding in market s would make a loss. Second, suppose that the mass of types in $\mathcal{E}(1)$ that bid in market s converges to a positive limit (i.e., $\sqrt{n}F_s^n(\mathcal{E}(1)|1)\to\infty$). In this case Property 2 implies that $b_s(z)=1$ for all finite z, i.e., the price in market s converges to one in state $V=1$. However now the logic we used above implies that $P_s^n \to 0$ in state $V=0$ (because otherwise the uninformed type would make a loss), $P_r^n \to 1$ in state $V=1$ (because otherwise all types in $\mathcal{E}(1)$ would select market r), and $P_r^n \to c$ in state $V=0$ (because no uninformed type selects market r). Hence, information is aggregated in both markets in this case also.

4.2. Pooling by pivotal types. We complete our discussion of the illustrative example by constructing an equilibrium in which there is pooling by pivotal types with uninformative prices. As we discussed in the previous subsection, pooling by pivotal types is not possible if types $\theta \in \mathcal{E}(1/2)$ are uninformed, i.e., if $\pi=1/2$. In order to construct an equilibrium where there is pooling by pivotal types, we alter the illustrative example by assuming that types $\theta \in \mathcal{E}(1/2)$ are pessimistic, i.e., $\pi<1/2$.

Example 4.1. Suppose that $0<\pi<c<\frac{1}{2}$, $\kappa_s<g$, and $\kappa_r>1-g$. There exists an $\epsilon>0$ such that, for all sufficiently large n, there is an equilibrium where all types $\theta \in \mathcal{E}(1)$

select market r and all types $\theta \in \mathcal{E}(1/2)$ submit the same pooling bid $b_p = c+\epsilon$ in market s. In this equilibrium, the price in market s is equal to b_p and the price in market r is equal to c with probability converging to one in both states.

In the equilibrium described above, types $\theta \in \mathcal{E}(1/2)$ are willing to submit the pooling bid because the probability of winning at the pooling bid is greater in state $V=1$. In fact, the posterior of a type $\theta \in \mathcal{E}(1/2)$, conditional on winning an object at the pooling bid, converges to 1/2 as the market grows large. Such types make a profit submitting the pooling bid because $b_p < 1/2$. Also, these types would neither want to outbid the pooling bid nor choose market r. If a type $\theta \in \mathcal{E}(1/2)$ outbids the pooling bid, then she always wins and hence her posterior, conditional on winning, is equal to $\pi < b_p$, i.e., she makes a loss. Similarly, such a type makes a loss if she chooses market r because $\pi < c$. No type $\theta \in \mathcal{E}(1)$ bids in market s because their payoff in market r is equal to $1-c$ which exceeds $1-b_p$. The construction of the equilibrium sequence described in the example above is in the online appendix.

4.3. General Signal Structures. In the proposition presented below, we allow for general signal structures that satisfy the assumptions made in Section 2 and we assume that the reserve price either exceeds 1/2 or is smaller than a particular cutoff $\bar{c}$. This cutoff is defined implicitly by the expression given below:

$$\bar{c}/(1-\bar{c}) := (1-\kappa_s-\kappa_r)^2/(1-\min\{\kappa_s,\kappa_r\}+1-\kappa_r-\kappa_s) > 0. \tag{4.2}$$

Under these assumptions, we argue that information aggregation fails due to a lack of competition in market r while information aggregation fails in market s because the pivotal types are arbitrarily close even though there is sufficient competition.

Proposition 4.2. *For any sequence of equilibria, the measure of types that submit a bid in market r satisfies $F_r(1|0) < F_r(1|1) \leq \kappa_r$, and the price in market r converges to c with probability one if $V=0$ and it converges to a binary random variable that is equal to c with probability $q>0$ and equal to one with the remaining probability if $V=1$. If $\kappa_s > \bar{\kappa}$, $\kappa_s+\kappa_r < 1$, and $c \notin [\bar{c},1/2]$, then the pivotal types in market s are arbitrarily close. If, in addition, $c < \bar{c}$, then $\lim\mathbb{E}[P_s^n|V=0] = \lim\mathbb{E}[P_r^n|V=0] = c$ and $\lim\mathbb{E}[P_r^n|V=1] = \lim\mathbb{E}[P_s^n|V=1]$, i.e., the expected prices are equal across states and markets.*

The proposition above shows that the limit outcome in market r is identical to the outcome we characterized for the illustrative example irrespective of the parameters of the model. If we further assume that $\kappa_s > \bar{\kappa}$ and $\kappa_s+\kappa_r < 1$, then the limit outcome depends on the reserve price. If $c > 1/2$, then the limit outcome in market s shares all the features of the outcome that we characterized for the illustrative example for the case where $c > 1/2$. On the other hand, if $c < \bar{c}$, then the limit outcome in market s shares all the features of the outcome that we characterized for the illustrative example for the case where $c < 1/2$. If the reserve price is not in the range covered by this proposition, then there are equilibria where information aggregation fails in market s because of pooling by pivotal types as in Example 4.1. If $\kappa_s + \kappa_r > 1$, then there are equilibria where information aggregation fails in market s also because of lack of competition as in market r. Finally, if $\kappa_s < \bar{\kappa}$, then information is aggregated in market s as shown by Theorem 3.1.

The characterization of the outcome in market r is provided in the appendix as part of Theorem 3.1's proof. We prove the results for market s by first noticing that there is sufficient competition in market s because the mass of types that select market r is at most κ_r and then arguing that pooling by pivotal types cannot be sustained if $c \notin [\bar{c}, 1/2]$ leaving arbitrarily close pivotal types as the only possible limit outcome. This argument is provided in the online appendix.

5. Discussion and Conclusion

The results that we presented in the paper argued that the price in a large, uniform-price, common-value auction may not aggregate all available information if bidders have access to an outside option that delivers state dependent payoffs. We showed how an outside option can hinder information aggregation by generating type-dependent selection into two auction markets. However, we studied only one example of an alternative market that provides an outside option that can hinder information aggregation. There are many other institutional configurations that could result in similar outcomes. For example, market r could instead be (1) A pay-as-you-bid (discriminatory price) auction as in Jackson and Kremer (2007), where all bidders that win an object from the auction pay their own bid, (2) An all-pay-auction as in Chi et al. (2019), or (3) A uniform-price auction where each bidder must pay a positive cost in order to submit a bid as in Murto and Valimaki (2014). The payoff distributions in these alternative specifications have similar properties to the payoff distribution in market r as described by Theorem 3.1: payoffs are negative in state $V = 0$ and positive in state $V = 1$. Our analysis suggests that information aggregation could be hindered by such market mechanisms also.

A. Appendix

A.1. Bidding Equilibria Suppose participation in market s is exogenously determined by a function $F_s(\cdot)$ that is absolutely continuous with respect to $F(\cdot)$. Given F_s, $\hat{\Gamma}(F_s)$ is the auction where each type θ is allowed to bid in the auction with probability $a(\theta)$ and is assigned a payoff equal to zero with the remaining probability $1 - a(\theta)$. The profile H is a bidding equilibrium if it is a Nash equilibrium of the auction $\hat{\Gamma}(F_s)$.

Let $\mathcal{E}(\theta')=\{\theta:l(\theta_i=\theta)=l(\theta_i=\theta')\}$. Each equivalence class $\mathcal{E}(\theta')$ is comprised of types who receive signals that generate the same posterior. If $\mathcal{E}(\theta')$ is not a singleton, then H may involve a range of bids given a signal in $\mathcal{E}(\theta')$. However, for any such H there is another strategy, which is pure and increasing on each $\mathcal{E}(\theta')$, such that this strategy yields the same payoff to the player, and is indistinguishable to any other player. Strategies which differ only in their representation over sets $\mathcal{E}(\theta')$ generate the same joint distribution over values, bids, and equilibrium prices. We choose a representation of H which is pure and nondecreasing over equivalence classes $\mathcal{E}(\theta')$.

The following lemma shows that the bids of the pivotal types determine the auction-clearing price of a sufficiently large auction.

Lemma A.1. *Suppose* $\lim \bar{F}_s^n(0|v)>\kappa_s$ *and let* $\underline{\theta}^n$ *denote the type such that* $F_s^n([\underline{\theta}^n,\theta_s^n(v)]|v)=\epsilon$ *and* $\underline{\theta}^n=0$ *if no such type exist. Similarly, let* $\bar{\theta}^n$ *denote the type such that* $F_s^n([\theta_s^n(v),\bar{\theta}^n]|v)=\epsilon$ *whenever such a type exists. For every* $\epsilon>0$*,* $\lim \Pr(P^n\in[b^n(\underline{\theta}^n),b^n(\bar{\theta}^n)]|V=v)=1$ *where* $b^n(0)=0$*. Conversely, if* $\lim \bar{F}_s^n(0|v)<\kappa_s$*, then* $\lim \Pr(P^n=0|V=v)=1$.

Proof. The LLN implies that $\lim \Pr(Y_s^n(k_s+1)\geq\underline{\theta}^n|V=v)=1$ for every $\epsilon>0$. However, if $Y_s^n(k_s+1)\geq\underline{\theta}^n$, then $P^n=b^n(Y_s^n(k_s+1))\geq b^n(\underline{\theta}^n)$ because b^n is nondecreasing by Lemma 2.1. Therefore, $\Pr(P^n\geq b^n(\underline{\theta}^n)|V=v)\geq\Pr(Y_s^n(k_s+1)\geq\underline{\theta}^n|V=v)$, and taking limits proves the first part of the claim. We establish $\lim \Pr(Y_s^n(k_s+1)\leq\bar{\theta}^n|V=v)=1$ using the same idea. If $\lim \bar{F}_s^n(0|v)<\kappa_s$, then $\lim \Pr(P^n=0|V=v)=1$ also follows directly from the LLN. □

A.1.1. *Pooling Calculations* Lemma 2.2 asserts that no pooling by pivotal types and distinct pivotal types are necessary and sufficient for information aggregation. In the next two subsections, we develop the intermediate results that we use to show that these two conditions are indeed necessary and sufficient for information aggregation. We use the results that we present here to determine when pooling by pivotal types is incompatible with equilibrium.

Given a strategy H, denote by $\Pr(b\,win|P^n=b,V=v,\theta)$ the conditional probability that bidder i wins an object with a bid equal to b given that the auction price is equal to b, the state is equal to v, and bidder i receives a signal equal to θ. Our assumptions that the signals are conditionally independent given V and that H is symmetric together imply that $\Pr(b\,win|P^n=b,V=v,\theta)=\Pr(b\,win|P^n=b,V=v)$. This is because once one conditions on the state, the individual signal of bidder i does not provide any additional information (conditional independence). Moreover, this probability is independent of the identity of the bidder that we consider because we focus on symmetric strategies.

Given a pooling bid b_p^n, let $\theta_p^n = \sup\{\theta : b^n(\theta) = b_p^n\}$, $\underline{\theta}_p^n = \inf\{\theta : b^n(\theta) = b_p^n\}$, and let $\lim\theta_p^n = \theta_p$ and $\lim\underline{\theta}_p^n = \underline{\theta}_p$ whenever these limits exist. Throughout this subsection, we focus on pools such that $\lim\sqrt{n}F_s^n([\underline{\theta}_p^n,\theta_p^n]|v) > 0$ for $v = 0,1$. The following lemma calculates $\Pr(b_p^n\, wins|P^n = b_p^n, V = v)$ for various case. The proof which involves lengthy statistical computations is provided in the online appendix.

Lemma A.2. *If* $\lim\Pr(P^n = b_p^n|v) > 0$*, then there is a constant* C *such that* $\Pr(b_p^n(\theta^n)\, lose|P^n = b_p^n, V = v) \geq C\frac{\max\{\kappa_s - F_s^n(\underline{\theta}_p^n|1), 1/\sqrt{n}\}}{F_s^n([\underline{\theta}_p^n,\theta_p^n]|1)}$ *for all sufficiently large* n*. If* $\lim F_s^n([\underline{\theta}_p^n,\theta_p^n]|v) > 0$*, then* $\lim\Pr(b_p^n\ win|P^n = b_p^n, V = v) = \lim\frac{\kappa_s - \bar{F}_s^n(\theta_p^n|v)}{F_s^n([\underline{\theta}_p^n,\theta_p^n]|v)}$*. If* $\lim\Pr(P^n \geq b_p^n|v = 0)$*, then* $\lim\Pr(b_p^n\, lose|P^n = b_p^n, V = v)/\frac{\overline{F}_s^n(\underline{\theta}_p^n|v)(1-\bar{F}_s(\underline{\theta}_p^n|v))}{nF_s^n([\underline{\theta}_p^n,\theta_p^n]|v)(\kappa_s - \bar{F}_s^n(\underline{\theta}_p^n|v))} = 1$.

Lemma A.3. *Fix a sequence of bidding equilibria* $\mathbf{H}$*. If* $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$ *or if* $F_s(1|1) \geq F_s(1|0)$ *and* $F_s(1|1) > \kappa_s$*, then there is no pooling by pivotal types.*

Proof. We will argue that if $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$, then pooling by pivotal types is incompatible with equilibrium. At the end of the proof we show that $F_s(1|1) \geq F_s(1|0)$ and $F_s(1|1) > \kappa_s$ imply $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$.

The fact that $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$ implies $\theta_s(1) > \theta_s(0)$ and $F_s(\theta_s(1)|0) > F_s(\theta_s(0)|0)$. Pooling by pivotal types implies that $F_s(\underline{\theta}_p|v) \leq F_s(\theta_s(0)|v) < F_s(\theta_s(1)|v) \leq F_s(\theta_p|v)$. We will show that pooling by pivotal types is incompatible with equilibrium behavior in the following three cases: (1) $F_s(\underline{\theta}_p|v) < F_s(\theta_s(0)|v)$ and $F_s(\theta_p|v) > F_s(\theta_s(1)|v)$; (2) $F_s(\theta_p|v) = F_s(\theta_s(1)|v)$ and $F_s(\underline{\theta}_p|v) < F_s(\theta_s(0)|v)$; and (3) $F_s(\underline{\theta}_p|v) = F_s(\theta_s(0)|v)$.

Case 1: $F_s(\underline{\theta}_p|v) < F_s(\theta_s(0)|v)$ and $F_s(\theta_p|v) > F_s(\theta_s(1)|v)$. For type θ_p bidding b_p instead of bidding slightly above the pooling bid is incentive-compatibility: $(1-b_p)l(\theta_p)\lim\Pr(P^n = b_p^n, b_p^n \quad win|V = 1) - b_p\lim\Pr(P^n = b_p^n, b_p^n \quad win|V = 0) \geq (1-b_p)l(\theta_p) - b_p$.Therefore,

$$\frac{b_p}{1-b_p} \geq l(\theta_p)\frac{\lim\Pr(P^n = b_p^n, b_p^n \quad loses|V = 1)}{\lim\Pr(P^n = b_p^n, b_p^n \quad loses|V = 0)}. \tag{A.1}$$

Pooling is individually rational for type $\underline{\theta}_p$: $(1-b_p)l(\underline{\theta}_p)\lim\Pr(P^n = b_p^n, b_p^n \quad win|V = 1) - b_p\lim\Pr(P^n = b_p^n, b_p^n \quad win|V = 0) \geq 0$. Therefore,

$$\frac{b_p}{1-b_p} \leq l(\underline{\theta}_p)\frac{\lim\Pr(P^n = b_p^n, b_p^n \quad wins|V = 1)}{\lim\Pr(P^n = b_p^n, b_p^n \quad wins|V = 0)}.$$

Combining the incentive compatibility and individual rationality constraints and substituting in using by Lemma A.2, we obtain

$$l(\underline{\theta}_p)\frac{F_s(\theta_p|1)-F_s(\theta_s(1)|1)}{F_s(\theta_p|0)-F_s(\theta_s(0)|0)}\geq l(\theta_p)\frac{F_s(\theta_s(1)|1)-F_s(\underline{\theta}_p|1)}{F_s(\theta_s(0)|0)-F_s(\underline{\theta}_p|0)},$$

which is not possible because $l(\underline{\theta}_p)\leq\frac{F_s(\theta_s(0)|1)-F_s(\underline{\theta}_p|1)}{F_s(\theta_s(0)|0)-F_s(\underline{\theta}_p|0)}<\frac{F_s(\theta_s(1)|1)-F_s(\underline{\theta}_p|1)}{F_s(\theta_s(0)|0)-F_s(\underline{\theta}_p|0)}$ and because $\frac{F_s(\theta_p|1)-F_s(\theta_s(1)|1)}{F_s(\theta_p|0)-F_s(\theta_s(0)|0)}<\frac{F_s(\theta_p|1)-F_s(\theta_s(1)|1)}{F_s(\theta_p|0)-F_s(\theta_s(1)|0)}\leq l(\theta_p)$ by MLRP.

Case 2: If $F_s(\theta_p|v)=F_s(\theta_s(1)|v)$ and $F_s(\underline{\theta}_p|v)<F_s(\theta_s(0)|v)$, then Lemma A.2 implies that $\lim\Pr(P^n=b_p^n,b_p^n\,wins|V=1)=0$ and $\lim\Pr(P^n=b_p^n,b_p^n\,wins|V=0)>0$. However, then pooling cannot be sustained by Lemma 7 and Corollary 3 in Pesendorfer and Swinkels (1997).

Case 3: If $F_s(\underline{\theta}_p|v)=F_s(\theta_s(0)|v)$, then Lemma A.2 implies that $\lim\Pr(P^n=b_p^n,b_p^n\,wins|V=0)=1$ and $\lim\Pr(P^n=b_p^n,b_p^n\,wins|V=1)<1$ again showing that pooling cannot be sustained by Lemma 7 and Corollary 3 in Pesendorfer and Swinkels (1997).

We conclude the proof by arguing that $F_s(1|1)\geq F_s(1|0)$ and $F_s(1|1)>\kappa_s$ together imply that $F_s(\theta_s(1)|1)>F_s(\theta_s(0)|1)$. On the way to a contradiction assume $F_s(\theta_s(1)|1)\leq F_s(\theta_s(0)|1)$. Note $F_s(1|1)>\kappa_s$ implies $0<F_s(\theta_s(1)|1)\leq F_s(\theta_s(0)|1)$. Our assumption $F_s(\theta_s(1)|1)\leq F_s(\theta_s(0)|1)$ and MLRP together imply that $1\geq\bar{F}_s(\theta_s(1)|1)/\bar{F}_s(\theta_s(1)|0)>F_s(\theta_s(1)|1)/F_s(\theta_s(1)|0)$. However, $F_s(1|v)=\bar{F}_s(\theta_s(1)|v)+F_s(\theta_s(1)|v)$, $\bar{F}_s(\theta_s(1)|1)\leq\bar{F}_s(\theta_s(1)|0)$, and $F_s(\theta_s(1)|1)<F_s(\theta_s(1)|0)$ together imply that $F_s(1|1)<F_s(1|0)$ leading to a contradiction. □

The following lemma shows that there cannot be a pool that occurs with positive probability in state $V=1$ and probability zero in state $V=0$ if the pivotal types are distinct.

Lemma A.4. *Fix a sequence of bidding equilibria* $\mathbf{H}$ *and assume* $\sqrt{n}(F_s^n(\theta_s^n(1)|1)-F_s^n(\theta_s^n(0)|1))\to\infty$. *There is no sequence of pooling bids* b_p^n *such that* $\lim\Pr(P^n=b_p^n|V=1)>0$ *and* $\lim\Pr(P^n\geq b_p^n|V=0)=0$.

Proof. We will show that $\lim\frac{\Pr(b^n\,lose|P^n=b_p^n,V=0)}{\Pr(b^n\,lose|P^n=b_p^n,V=1)}=0$ which implies that pooling cannot be sustained for sufficiently large n by Lemma 7 and Corollary 3 in Pesendorfer and Swinkels (1997). Lemma A.2 gives that

$$\lim\frac{\Pr(b^n\,lose|P^n=b_p^n,0)}{\Pr(b^n\,lose|P^n=b_p^n,1)}\leq\lim\frac{F_s^n([\underline{\theta}_p^n,\theta_p^n]|1)}{F_s^n([\underline{\theta}_p^n,\theta_p^n]|0)}\frac{C\overline{F}_s^n(\underline{\theta}_p^n|0)(1-\bar{F}_s(\underline{\theta}_p^n|0))}{n(\kappa_s-\bar{F}_s^n(\underline{\theta}_p^n|0))\max\{\kappa_s-F_s^n(\underline{\theta}_p^n|1),1/\sqrt{n}\}}$$

where $C\in(0,\infty)$. However, $F_s^n([\underline{\theta}_p^n,\theta_p^n]|1)/F_s^n([\underline{\theta}_p^n,\theta_p^n]|0)\leq 1/\eta$ by Lemma A.5, $n\max\{\kappa_s-F_s^n(\underline{\theta}_p^n|1),1/\sqrt{n}\}\geq\sqrt{n}$, $\overline{F}_s^n(\underline{\theta}_p^n|0)(1-\bar{F}_s(\underline{\theta}_p^n|0))\leq 1$, and $\lim\sqrt{n}(\kappa_s-\bar{F}_s^n(\underline{\theta}_p^n|0))=\infty$ (because $\lim \Pr(P^n \geq b_p^n | V=0)=0)$. Therefore, $\lim \Pr(b^n\, lose | P^n = b_p^n, 0)/\Pr(b^n\, lose | P^n = b_p^n, 1) \leq \lim 1/(C\eta\sqrt{n}(\kappa_s - \bar{F}_s^n(\underline{\theta}_p^n|0)))=0$. $\square$

A.1.2. *Information content of being pivotal.* In this subsection, we provide bounds for the ratio $l(Y^n(k_s+1)=\theta^n)=\Pr(Y^n(k_s+1)=\theta^n|V=1)/\Pr(Y^n(k_s+1)=\theta^n|V=0)$, i.e., the information content of the event of being pivotal. We will then use these bounds to show that distinct pivotal types are a necessary condition for information aggregation. Also, we will use the results in this subsection together with the results in the previous subsection to argue that distinct pivotal types and no pooling by pivotal types together imply information aggregation (Lemma 2.2). The results we present below show that the event of being pivotal provides only bounded amounts of information for the types that set the price if the pivotal types are arbitrarily close.

We begin with the following lemma that outlines the implication of our assumption that there are no arbitrarily informative signals.

Lemma A.5. *For any interval* $I \subset [0,1]$*,* $F_s^n(I|V=1) \in \left[\eta F_s^n(I|V=0), \frac{F_s^n(I|V=0)}{\eta}\right]$*. Therefore,* $\sqrt{n}(F_s^n(\theta_s^n(1)|0) - F_s^n(\theta_s^n(0)|0)) < \infty$ *iff* $\sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| < \infty$*.*

Proof. To see this, note $F_s^n(I|1)=\int_I a(\theta)f(\theta|1)d\theta = \int_I a(\theta)f(\theta|0)l(\theta)d\theta$ and $\eta F_s^n(I|0) = \eta\int_I a(\theta)f(\theta|0)d\theta \leq \int_I a(\theta)f(\theta|0)l(\theta)d\theta \leq \frac{1}{\eta}\int_I a(\theta)f(\theta|0)d\theta = \frac{1}{\eta}F_s^n(I|0)$ because $l(\theta) \in (\eta, 1/\eta)$ for $\theta \in [0,1]$. $\square$

For any $\theta \in [0,1]$ and $v=0,1$ define

$$z_v^n(\theta) := \frac{k_s - (n-1)\bar{F}_s^n(\theta|v)}{\sqrt{(n-1)\kappa_s(1-\kappa_s)}}$$

which measures the distance between type θ and the pivotal type $\theta_s^n(v)$ in terms of standard deviations.

The probability that a particular type θ is pivotal (i.e., $Y_s^n(k_s+1)=\theta$) can be approximated using the central limit theorem. If $\lim \frac{n\kappa_s - n\bar{F}_s^n(\theta|v)}{\sqrt{n\kappa_s(1-\kappa_s)}} = a$, then $Bi(k_s; n, \bar{F}_s^n(\theta|v)) \to \Phi(a)$ where Bi and Φ denote the binomial and standard normal cumulative distributions, respectively. Moreover, if we let $p=\bar{F}_s^n(\theta|v)$, then

$$bi(k_s;n,p) = \binom{n}{k_s} p^{k_s}(1-p)^{n-k_s} = \frac{1+\delta_n(p)}{\sqrt{2\pi n\kappa_s(1-\kappa_s)}}\phi\left(\frac{k_s - np}{\sqrt{\kappa_s(1-\kappa_s)n}}\right) \tag{A.2}$$

where bi and ϕ denote the binomial and standard normal densities, respectively; and $\lim_{n\to\infty}\sup_{p:|np-k_s|<n^t}\delta_n(p)=0$ for $t<2/3$ (see Lesigne (2005, Proposition 8.2)). In the following two lemmata, we use these convergence results and show that if the price is set by a type θ that is within finitely many standard deviations of both pivotal types, then the information that this type gets from being pivotal is bounded.

Lemma A.6. *Pick a sequence of types $\{\theta^n\}$ that bid in market s. Assume that $\lim z_v^n(\theta^n)=z_v$ for $v=0,1$ and $\lim l(\theta^n)=\rho$. For any $\delta>0$, there exists an N such that for all $n>N$ we have $\rho(1-\delta)\phi(z_1)/\phi(z_0)\le l(Y^n(k_s+1)=\theta^n)\le\rho(1+\delta)\phi(z_1)/\phi(z_0)$. Therefore, $l(Y^n(k_s+1)=\theta^n)\to\phi(z_1)/\phi(z_0)\rho$.*

Proof. A direct computation shows that $l(Y^n(k_s+1)=\theta^n)=l(\theta^n)\frac{bi(k_s;n-1,\bar{F}_s^n(\theta^n|1))}{bi(k_s;n-1,\bar{F}_s^n(\theta^n|0))}$. Eq. (A.2) implies that for any $\delta>0$, there exists an N such that $(1-\delta)\phi(z_1^n(\theta^n))/\phi(z_0^n(\theta^n))\le bi(k_s;n-1,\bar{F}_s^n(\theta^n|1))/bi\left(k_s;n-1,\bar{F}_s^n(\theta^n|0)\right)\le(1+\delta)\phi(z_1^n(\theta^n))/\phi(z_0^n(\theta^n))$ for all $n>N$. Our assumption that $\lim z_v^n(\theta^n)=z_v$ and $k_s/(n-1)\to\kappa_s$ together establish that $\lim\sqrt{n}|\bar{F}_s^n(\theta^n|v)-\kappa_s|<\infty$ for $v=0,1$. The fact that $\phi(z_v^n(\theta))$ is a continuous functions of θ implies that for any $\delta>0$, there exists an N such that for all $n>N$ we have $\rho(1-\delta)\phi(z_1)/\phi(z_0)\le l(Y^n(k_s+1)=\theta^n)\le\rho(1+\delta)\phi(z_1)/\phi(z_0)$. □

Lemma A.7. *Assume $\lim\sqrt{n}|F_s^n(\theta_s^n(1)|1)-F_s^n(\theta_s^n(0)|1)|=x<\infty$ and $\lim\sqrt{n}(\kappa_s-\bar{F}_s^n(0|0))=-\infty$. Suppose θ_y^n is a the type such that $F_s^n([\theta_y^n,\theta_s^n(0)]|0)=\sqrt{\kappa_s(1-\kappa_s)/n}$. For any $\delta>0$, there exists an N such that for all $n>N$ and for any interval $[a,b]\subset[\theta_y^n,\theta_s^n(0)]$ such that $F_s^n([a,b]\,|0)>0$ we have $\eta\,(1-\delta)\,\phi(x+y/\eta)/\phi\,(0)\le l(Y^n(k_s+1)\in[a,b])\le(1+\delta)\phi(0)/\eta\phi(y)$*

Proof. Suppose, without loss of generality, that $\lim\frac{\sqrt{n}(F_s^n(\theta_s^n(1)|1)-F_s^n(\theta_s^n(0)|1))}{\sqrt{\kappa_s(1-\kappa_s)}}\ge0$. Note that if $\lim\sqrt{n}(\kappa_s-\bar{F}_s^n(0|0))=-\infty$, then $\lim\sqrt{n}F_s(\theta_s^n(0)|0)=\infty$ and the interval $\left[\theta_y^n,\theta_s^n(0)\right]$ is well defined for all sufficiently large n. For any sequence $\{\theta^n\}$ such that $\theta^n\in[\theta_y^n,\theta_s^n(0)]$ for every n, we have $\lim z_v^n(\theta^n)=\lim\sqrt{n}(\kappa_s-\bar{F}_s^n(\theta^n|v))/\sqrt{\kappa_s(1-\kappa_s)}$. Also, $l(\theta)\in[\eta,1/\eta]$ (because there are no arbitrarily informative signals), $\lim z_1^n(\theta^n)\in[-x-y/\eta,0]$, and $\lim z_0^n(\theta^n)\in[-y,0]$. Therefore, Lemma A.6 implies that for any $\delta>0$, there exists an N such that for all $n>N$ and any $\theta\in[\theta_y^n,\theta_s^n(0)]$

$$(1-\delta)\frac{\phi(x+y/\eta)}{\phi(0)}\le\frac{bi\left(k_s;n-1,\bar{F}_s^n(\theta|1)\right)}{bi\left(k_s;n-1,\bar{F}_s^n(\theta|0)\right)}\le(1+\delta)\frac{\phi(0)}{\phi(y)}.$$

Thus using the fact that $l(\theta)\in[\eta,1/\eta]$, we conclude that $\eta(1-\delta)\phi(x+y/\eta)/\phi(0)\le l(Y^n(k_s+1)\in[a,b])\le(1+\delta)\phi(0)/\eta\phi(y)$ for any interval $[a,b]\subset[\theta_y^n,\theta^n(0)]$. □

A.1.3. *Proof of Information Aggregation Lemma 2.2.*

Proof of Lemma 2.2. First we argue that if $\mathbf{H}$ aggregates information, then there is no pooling by pivotal types and the pivotal types are distinct. Note that if there is pooling by pivotal types, then $\mathbf{H}$ does not aggregate information by definition.[18] Let $\sigma := \sqrt{\kappa_s(1-\kappa_s)}$ and recall that $\bar{F}^n_s(0|0)$ is the fraction of types who bid in market s in state 0. We will argue that if $\mathbf{H}$ aggregates information, then the pivotal types are distinct $(\lim\sqrt{n}|F^n_s(\theta^n_s(1)|1)-F^n_s(\theta^n_s(0)|1)|=\infty)$. Suppose the pivotal types are arbitrarily close $(\lim\sqrt{n}|F^n_s(\theta^n_s(1)|1)-F^n_s(\theta^n_s(0)|1)|<\infty)$. In the next two claims, we will show 1) If the number of objects exceeds the number of bidders with positive probability in state $V=0$ (i.e., if $\lim\sqrt{n}(\kappa_s - \bar{F}^n_s(0|0)) > -\infty$), then $\mathbf{H}$ does not aggregate information; and 2) If the number of bidders exceeds the number of objects with probability one in state $V=0$ (i.e., if $\lim\sqrt{n}(\kappa_s-\bar{F}^n_s(0|0))=-\infty$), then $\mathbf{H}$ does not aggregate information. Therefore, we will conclude that if the pivotal types are arbitrarily close, then $\mathbf{H}$ does not aggregate information establishing our claim.

Claim A.1. If $\lim\sqrt{n}|F^n_s(\theta^n_s(1)|1)-F^n_s(\theta^n_s(0)|1)|<\infty$ and $\lim\sqrt{n}(\kappa_s-\bar{F}^n_s(0|0))>-\infty$, then $\mathbf{H}$ does not aggregate information.

Proof. Suppose $\sqrt{n}|F^n_s(\theta^n_s(1)|1)-F^n_s(\theta^n_s(0)|1)| < \infty$. We will show that the price is equal to zero with strictly positive probability in both states and therefore $\mathbf{H}$ does not aggregate information. Suppose that $\sqrt{n}(\kappa_s-\bar{F}^n_s(0|0))/\sigma \to x > -\infty$ where x is possibly equal to $+\infty$. The central limit theorem implies that the number of goods in the auction exceeds the number of bidders with positive probability if $V=0$, and $\lim_n \Pr(Y^n_s(k_s+1)=0|V=0)=\Phi(x)>0$.

Below we argue that $\sqrt{n}|F^n_s(\theta^n_s(1)|1)-F^n_s(\theta^n_s(0)|1)|<\infty$ and $\sqrt{n}(\kappa_s-\bar{F}^n_s(0|0))/\sigma \to x > -\infty$ together imply that $\sqrt{n}(\kappa_s - \bar{F}^n_s(0|1))/\sigma \to x' > -\infty$. But if $\sqrt{n}(\kappa_s - \bar{F}^n_s(0|1))\sigma \to x' > -\infty$, then applying the central limit theorem once again we find $\lim_n \Pr(Y^n_s(k_s+1)=0|V=1)=\Phi(x')>0$ and therefore $\lim_n \Pr(P^n=0|V=1)\geq\Phi(x')>0$. However, $\lim_n \Pr(P^n=0|V=v)>0$ for $v=0,1$ and $\lim_n l(P^n=0)=\Phi(x')/\Phi(x)\in(0,\infty)$ contradicts that $\mathbf{H}$ aggregates information.

We argue that $\sqrt{n}(\kappa_s-\bar{F}^n_s(0|0))/\sigma = \sqrt{n}\left(\kappa_s-\left(F^n_s(\theta^n_s(0)|0)+\bar{F}^n_s(\theta^n_s(0)|0)\right)\right)/\sigma \to x > -\infty$ implies $\lim\sqrt{n}F_s(\theta^n_s(0)|0) < \infty$ and therefore $\lim\sqrt{n}F_s(\theta^n_s(0)|1) < \infty$. By definition we have $\bar{F}^n_s(\theta^n_s(0)|0)=\kappa_s$ if $\kappa_s \leq \bar{F}^n_s(0|0)$ and $\theta^n_s(0)=0$ otherwise. Therefore, $\sqrt{n}(\kappa_s-\bar{F}^n_s(0|0))=-\infty$ if and only if $\lim\sqrt{n}F_s(\theta^n_s(0)|0)=\infty$. Hence, $\sqrt{n}(\kappa_s-\bar{F}^n_s(0|0))/\sigma \to x > -\infty$ implies that $\lim\sqrt{n}F_s(\theta^n_s(0)|0)<\infty$ and hence $\lim\sqrt{n}F_s(\theta^n_s(0)|1)<\infty$ by Lemma A.5.

We now show that $\lim\sqrt{n}(\kappa_s-\bar{F}^n_s(0|0))=x>-\infty$ implies $\lim\sqrt{n}(\kappa_s-\bar{F}^n_s(0|1))>-\infty$. We argued in the previous paragraph that $\sqrt{n}(\kappa_s - \bar{F}^n_s(0|1))/\sigma \to -\infty$ if and only if $\lim\sqrt{n}F_s(\theta^n_s(1)|1) = \infty$. However, if $\lim\sqrt{n}F_s(\theta^n_s(1)|1) = \infty$, then

[18] This is because $\lim\Pr\left(P^n=b^n_p|V=v\right)>0$ for $v=0,1$, i.e., the auction price is equal to the pooling bid with strictly positive probability in both states.

$\lim\sqrt{n}F_s^n([\theta_s^n(0),\theta_s^n(1)]|1)=\infty$ because $\lim\sqrt{n}F_s(\theta_s^n(0)|0)<\infty$ and because $\lim\sqrt{n}F_s(\theta_s^n(0)|1)<\infty$. But this contradicts $\sqrt{n}|F_s^n(\theta_s^n(1)|1)-F_s^n(\theta_s^n(0)|1)|<\infty$. Hence, $\sqrt{n}(\kappa_s-\bar{F}_s^n(0|1))/\sigma\to x'$ for some $x'>-\infty$ which is possibly equal to $+\infty$. □

We now turn to the case where $\lim\sqrt{n}(\kappa_s-\bar{F}_s^n(0|0))=-\infty$. Pick any $y>0$ and let θ_y^n denote the type such that

$$F_s^n([\theta_y^n,\theta_s^n(0)]|0)=\sigma y/\sqrt{n} \tag{A.3}$$

when such a type exists. Observe that $\theta_y^n<\theta_{2y/3}^n<\theta_{y/3}^n<\theta^n(0)$ and $F_s^n([\theta_{2y/3}^n,\theta_{y/3}^n]|0)=\sigma y/3/\sqrt{n}$ by the definition of these types given in Eq. (A.3). Let $A^n:=\{p:p=b^n(\theta),\theta\in[\theta_{2y/3}^n,\theta_{y/3}^n]\}$. The central limit theorem implies that $\lim\Pr(Y^n(k+1)\in[\theta_{2y/3}^n,\theta_{y/3}^n]|V)=0=\Phi(2y/3)-\Phi(y/3)>0$. Also, $\Pr(P^n\in A^n|V=0)\geq\Pr(Y^n(k+1)\in[\theta_{2y/3}^n,\theta_{y/3}^n]|V)=0$ because $P^n=b^n(Y^n(k+1))$. The inequality above does not necessarily hold as an equality because types other than those $[\theta_{2y/3}^n,\theta_{y/3}^n]$ may also choose a bid in A^n. Lemma A.7 implies that $\lim\Pr(Y^n(k+1)\in[\theta_{2y/3}^n,\theta_{y/3}^n]|V=1)\geq\frac{\phi(x+y/\eta)}{\phi(0)}\eta(\Phi(2y/3)-\Phi(y/3))>0$. Therefore $\lim\Pr(P^n\in A^n|V=1)\geq\frac{\phi(x+y/\eta)}{\phi(0)}\eta(\Phi(2y/3)-\Phi(y/3))>0$.

Claim A.2. If $\lim\sqrt{n}|F_s^n(\theta_s^n(1)|1)-F_s^n(\theta_s^n(0)|1)|=x<\infty$ and $\lim\sqrt{n}(\kappa_s-\bar{F}_s^n(0|0))=-\infty$, then **H** does not aggregate information.

Proof. We will argue that there exists an $\epsilon>0$ such that $l(P^n=p)\in(\epsilon,1/\epsilon)$ for any $p\in A^n$ and any n sufficiently large. However, this together with the facts that $\Pr(P^n\in A^n|V=1)>0$ and $\Pr(P^n\in A^n|V=0)>0$ imply that **H** does not aggregate information.

Pick any $\delta>0$. For any $\theta^*\in[\theta_{2y/3}^n,\theta_{y/3}^n]$ that bids in market s with positive probability, we have either 1) $\{\theta:b^n(\theta)=b^n(\theta^*)\}\subset[\theta_y^n,\theta^n(0)]$ or 2) $\{\theta:b^n(\theta)=b^n(\theta^*)\}\not\subseteq[\theta_y^n,\theta^n(0)]$. Moreover, the fact that the bidding function is monotone implies that the set $\{\theta:b^n(\theta)=b^n(\theta^*)\}$ is either a singleton or an interval.

If $\{\theta:b^n(\theta)=b^n(\theta^*)\}\subset[\theta_y^n,\theta^n(0)]$, then Lemma A.7 implies that

$$(1-\delta)\frac{\phi(x+y/\eta)}{\phi(0)}\eta\leq l(Y^n(k+1)\in\{\theta:b^n(\theta)=b^n(\theta^*)\})\leq(1+\delta)\frac{\phi(0)}{\phi(y)}\frac{1}{\eta}$$

for all $n>N(\delta)$.[19] Therefore, $\eta(1-\delta)\phi(x+y/\eta)/\phi(0)\leq l(P^n=b^n(\theta^*))\leq(1+\delta)\phi(0)/\phi(y)\eta$ for all $n>N(\delta)$.

If, on the other hand, $\{\theta:b^n(\theta)=b^n(\theta^*)\}\not\subseteq[\theta_y^n,\theta^n(0)]$, then either $[\theta_y^n,\theta_{2y/3}^n]\subset\{\theta:b^n(\theta)=b^n(\theta^*)\}$ or $[\theta_{y/3}^n,\theta^n(0)]\subset\{\theta:b^n(\theta)=b^n(\theta^*)\}$ because the set $\{\theta:b^n(\theta)=b^n(\theta^*)\}$ is an interval that extends beyond $[\theta_y^n,\theta^n(0)]$. Therefore, $\Pr(P^n=b^n(\theta^*)|V=v)\geq$

[19] Observe that $N(\delta)$ is independent of θ^* and the set $[\theta_{2y/3}^n,\theta_{y/3}^n]$.

$\Pr(Y^n(k+1) \in [\theta^n_{y/3}, \theta^n(0)]|V=v)$ or $\Pr(P^n = b^n(\theta^*)|V=v) \geq \Pr(Y^n(k+1) \in [\theta^n_y, \theta^n_{2y/3}]|V=v)$. The central limit theorem implies that $(1-\delta)(\Phi(y/3)-1/2) \leq \Pr(Y^n(k+1) \in [\theta^n_{y/3}, \theta^n(0)]|V=0)$ and$(1-\delta)(\Phi(y)-\Phi(2y/3)) \leq \Pr(Y^n(k+1) \in [\theta^n_y, \theta^n_{2y/3}]|V=0)$ for all for all $n > N(\delta)$. Moreover, Lemma A.7 implies that $(1-\delta)\phi(x+y/\eta)\eta(\Phi(y/3)-1/2)/\phi(0) \leq \Pr(Y^n(k+1) \in [\theta^n_{y/3}, \theta^n(0)]|V=1)$ and $(1-\delta)\phi(x+y/\eta)\eta(\Phi(y)-\Phi(2y/3))/\phi(0) \leq \Pr(Y^n(k+1) \in [\theta^n_y, \theta^n_{2y/3}]|V=1)$ for all $n > N(\delta)$. Therefore, $(1-\delta)\phi(x+y/\eta)C\eta/\phi(0) \leq l(P^n = b^n(\theta^*)) \leq 1/(1-\delta)C$ for all for all $n > N(\delta)$ where $C = \min\{\Phi(y/3)-1/2, \Phi(y)-\Phi(2y/3)\}$. Hence picking ϵ such that $\epsilon < \phi(x+y/\eta)\eta C/\phi(0)$, $\epsilon < C$ and $1/\epsilon > \phi(0)/\phi(y)\eta$ establishes that $\mathbf{H}$ does not aggregate information. ∎

We now argue that if there is no pooling by pivotal types and if the pivotal types are distinct, then information is aggregated along a sequence $\mathbf{H}$. Denote by $v \in \{0,1\}$ the state where the pivotal type is largest and by v' the other state. Our assumption that the pivotal types are distinct implies that $\sqrt{n}F^n_s([\theta^n_s(v'), \theta^n_s(v)]|0) \to \infty$. For any $\epsilon \in (0,1/2)$ define

$$\bar{\theta}^n_\epsilon := \min\{\theta : \Pr(Y^n_s(k+1) \leq \theta|V=v)\} = \epsilon, \tag{A.4}$$

$\underline{\theta}^n_\epsilon := \max\{\theta : \Pr(Y^n_s(k+1) \geq \theta|V=v') = \epsilon\}$, and $b^n_\epsilon := (b^n(\underline{\theta}^n_\epsilon) + b^n(\bar{\theta}^n_\epsilon))/2$. These definitions imply that $\theta^n_s(v') < \underline{\theta}^n_\epsilon < \bar{\theta}^n_\epsilon < \theta^n_s(v)$ for sufficiently large n. This is because $\lim \sqrt{n}F^n_s\left(\left[\bar{\theta}^n_\epsilon, \theta^n_s(v)\right]|V=v\right) \in (0,\infty)$ and $\lim \sqrt{n}F^n_s([\theta^n_s(v'), \underline{\theta}^n_\epsilon]|V=v') \in (0,\infty)$ by the LLN (or a simple application of Chebyshev's inequality) and $\sqrt{n}F^n_s([\theta^n_s(v'), \theta^n_s(v)]|0) \to \infty$.

We prove the result through the three claims given below. We first argue that $\Pr(Y^n_s(k+1) \leq \underline{\theta}^n_\epsilon|v) \to 0$ and $\Pr(Y^n_s(k+1) \geq \bar{\theta}^n_\epsilon|v') \to 0$ (Claim A.3). We then show that the types $\underline{\theta}^n_\epsilon$ and $\bar{\theta}^n_\epsilon$ submit distinct bids and therefore $b^n(\underline{\theta}^n_\epsilon) < b^n(\bar{\theta}^n_\epsilon)$ (Claim A.4). We complete the proof by showing that the bid distribution is state v lies above b^n_ϵ and the bid distribution in state v' lies below b^n_ϵ with probability converging to one, i.e., b^n_ϵ separates the two bid distributions (Claim A.5).

Claim A.3. If $\sqrt{n}F^n_s([\theta^n_s(v'), \theta^n_s(v)]|0) \to \infty$, then $\Pr(Y^n_s(k+1) \leq \underline{\theta}^n_\epsilon|v) \to 0$ and $\Pr(Y^n_s(k+1) \geq \bar{\theta}^n_\epsilon|v') \to 0$.

Proof. Note $\sqrt{n}F^n_s([\underline{\theta}^n_\epsilon, \bar{\theta}^n_\epsilon]|0) \to \infty$ because $\lim \sqrt{n}F^n_s([\theta^n_s(v'), \theta^n_s(v)]|0) = \lim \sqrt{n}(F^n_s([\theta^n_s(v'), \underline{\theta}^n_\epsilon]|0) + F^n_s([\underline{\theta}^n_\epsilon, \bar{\theta}^n_\epsilon]|0) + F^n_s([\bar{\theta}^n_\epsilon, \theta^n_s(v)]|0)$, $\lim \sqrt{n}F^n_s([\theta^n_s(v'), \theta^n_s(v)]|0) = \infty$, $\lim \sqrt{n}F^n_s([\bar{\theta}^n_\epsilon, \theta^n_s(v)]|0) \in (0,\infty)$ and $\lim \sqrt{n}F^n_s([\theta^n_s(v'), \underline{\theta}^n_\epsilon]|0) \in (0,\infty)$. Moreover, $\sqrt{n}F^n_s([\underline{\theta}^n_\epsilon, \theta^n_s(v)]|v) \to \infty$ and $\sqrt{n}F^n_s([\theta^n_s(v'), \bar{\theta}^n_\epsilon]|v') \to \infty$ follow immediately from $\sqrt{n}F^n_s([\underline{\theta}^n_\epsilon, \theta^n_s(v)]|v) \geq \sqrt{n}F^n_s([\underline{\theta}^n_\epsilon, \bar{\theta}^n_\epsilon]|v)$ and $\sqrt{n}F^n_s([\theta^n_s(v'), \bar{\theta}^n_\epsilon]|v') \geq \sqrt{n}F^n_s([\underline{\theta}^n_\epsilon, \bar{\theta}^n_\epsilon]|v')$. Finally, the LLN implies that $\Pr(Y^n_s(k+1) \leq \underline{\theta}^n_\epsilon|v) \to 0$ and $\Pr(Y^n_s(k+1) \geq \bar{\theta}^n_\epsilon|v') \to 0$. ∎

Claim A.4. If the pivotal types are distinct and there is no pooling by pivotal types, then $b^n(\underline{\theta}^n_\epsilon) < b^n(\bar{\theta}^n_\epsilon)$ for all sufficiently large n.

Proof. Monotonicity implies $b^n(\underline{\theta}^n_\epsilon) \leq b(\bar{\theta}^n_\epsilon)$. Suppose $b^{n_k}(\underline{\theta}^{n_k}_\epsilon) = b^n(\bar{\theta}^{n_k}_\epsilon) = b^{n_k}_p$ for all n_k along a subsequence. Then, $\lim \Pr(P^{n_k} = b^{n_k}_p | V = v) \geq \epsilon > 0$ for each $v = 0,1$ by Claim A.3. However, this means that there is pooling by pivotal types contradicting the assumption of the claim. □

Claim A.5. If the pivotal types are distinct and there is no pooling by pivotal types, then **H** aggregates information.

Proof. Fix any $\epsilon \in (0,1/2)$. Claim A.4 implies $b^n(\underline{\theta}^n_\epsilon) < b^n_\epsilon < b^n(\bar{\theta}^n_\epsilon)$ for sufficiently large n. Given this definition, we have $\Pr(P^n \leq b^n_\epsilon | V = v) \leq \epsilon$ and $\lim \Pr(P^n \geq b^n_\epsilon | V = v') \leq \epsilon$. Moreover,

$$\int_{p<b^n_\epsilon} \frac{\Pr(P^n = p|V = v)}{\Pr(P^n = p|V = v')} \Pr(P^n = p|V = v') dp = \int_{p<b^n_\epsilon} \Pr(P^n = p|V = v) dp \leq \epsilon$$

Therefore, $\Pr(P^n \in \{p < b^n_\epsilon : \frac{\Pr(V=v|P^n=p)}{\Pr(V=v'|P^n=p)} > \sqrt{\epsilon}\} | V = v') \leq \sqrt{\epsilon}$. Hence,

$$\lim \Pr(P^n \in \{\frac{\Pr(V=v|P^n=p)}{\Pr(V=v'|P^n=p)} > \sqrt{\epsilon}\} | V = v') \leq \sqrt{\epsilon} + \lim \Pr(P^n \geq b^n_\epsilon | V = v') < 2\sqrt{\epsilon}.$$

Finally, for any $\epsilon' > \sqrt{\epsilon}$ we find $\lim \Pr(P^n \in \{\frac{\Pr(V=v|P^n=p)}{\Pr(V=v'|P^n=p)} > \epsilon'\} | V = v') < 2\sqrt{\epsilon}$. Because, ϵ is arbitrary, we conclude that $\lim \Pr(P^n \in \{\frac{\Pr(V=v|P^n=p)}{\Pr(V=v'|P^n=p)} > \epsilon'\} | V = v') = 0$ and a symmetric argument establishes the result for $V = v$. □

□

Lemma A.8 (Price converges to value)**.** *If an equilibrium sequence aggregates information and* $\lim \mathbb{E}[P^n] > 0$*, then* P^n *converges in probability to* V*.*

Proof. We prove the result through two claims. In the first claim we show that if information is aggregated and the expected price is positive, then the pivotal types must be ordered. In the second claim we show that if the pivotal types are ordered, then price must converge to value.

Claim A.6. If **H** aggregates information and $\lim \mathbb{E}[P^n] > 0$, then $\sqrt{n}(F^n_s(\theta^n_s(1)|1) - F^n_s(\theta^n_s(0)|1)) \to \infty$.

Proof. If **H** aggregates information, then $\sqrt{n}|F^n_s(\theta^n_s(1)|1) - F^n_s(\theta^n_s(0)|1)| \to \infty$ and there is no pooling by pivotal types by Lemma 2.2. Pick a subsequence (abusing notation, we omit the relabeling of this subsequence) and assume, contrary to the claim that

$\sqrt{n}(F_s^n(\theta_s^n(0)|1) - F_s^n(\theta_s^n(1)|1)) \to \infty$ along this subsequence. Moreover, suppose that $\lim \mathbb{E}[P^n|V=0]$ and $\lim \mathbb{E}[P^n|V=1]$ exist along this subsequence.

Recall the definition of b_ϵ^n given by Eq. A.4. The facts that **H** aggregates information and $\sqrt{n}(F_s^n(\theta_s^n(0)|1) - F_s^n(\theta_s^n(1)|1)) \to \infty$ together imply that $\lim \mathbb{E}[P^n|V=0] \geq \lim \mathbb{E}[P^n] \geq \lim \mathbb{E}[P^n|V=1]$ and in particular $\lim \mathbb{E}[P^n|V=0] \geq \lim \mathbb{E}[P^n] > 0$. This is because $\mathbb{E}[P^n|V=0] \geq (1-\epsilon)b_\epsilon^n$ and $\mathbb{E}[P^n|V=1\} \leq (1-\epsilon)b_\epsilon^n + \epsilon$ together imply that $\mathbb{E}[P^n|V=0] + \epsilon \geq \mathbb{E}[P^n|V=1]$ for each ϵ. Consider any type that submits a bid equal to b_ϵ^n. We have $\Pr(P^n < b_\epsilon^n|V=1) \geq 1-\epsilon$ and $\Pr(P^n > b_\epsilon^n|V=0) \geq 1-\epsilon$ by definition. Therefore, $u(b_\epsilon^n|\theta) \geq \Pr(V=1|\theta)(1-\epsilon)(1-\mathbb{E}[P^n|V=1]) - \Pr(V=0|\theta)\epsilon$ for any type θ. As ϵ is arbitrary, we find $\lim u(b^n(\theta)|\theta) \geq \Pr(V=1|\theta)(1-\lim \mathbb{E}[P^n|V=1])$ for each θ.

For a given $\epsilon \in (0, \kappa_s)$, pick any type $\theta > \theta_s(0) \geq \theta_s(1)$ such that $\bar{F}_s^n(\theta|0) < \epsilon$. Note that $\lim \Pr(P^n \leq b^n(\theta)|V=v) = 1$ for $v = 0, 1$. This type wins with probability at least $\kappa_s - \epsilon$ in state $V = 0$. This is because if the type θ bids in a pool with $\theta_s(0)$, then the probability of winning is at least $\kappa_s - \epsilon$ in state $V = 0$ by Lemma A.2. Otherwise, this type wins with probability one in both states. Therefore, $\lim u(b^n(\theta)|\theta) \leq \Pr(V=1|\theta)(1-\lim \mathbb{E}[P^n|V=1]) - (\kappa_s - \epsilon)\Pr(V=0|\theta)\lim \mathbb{E}[P^n|V=0] < \Pr(V=1|\theta)(1-\lim \mathbb{E}[P^n|V=1])$ leading to a contradiction. □

Claim A.7. Suppose **H** aggregates information. If $\lim \mathbb{E}[P^n] > 0$ or if $\sqrt{n}(F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)) \to \infty$, then $\lim \mathbb{E}[P^n|V=0] = 0$ and $\lim \mathbb{E}[P^n|V=1] = 1$.

Proof. Information aggregation and $\lim \mathbb{E}[P^n] > 0$ together imply that $\sqrt{n}(F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)) \to \infty$ by the previous claim. Assume to the contrary that $\lim \mathbb{E}[P^n|V = 0] > 0$ along a convergent subsequence. There are two cases to consider: 1) There is an $\epsilon < \eta \lim \mathbb{E}[P^n|V=0]$ and a subsequence such that $\Pr(P^n = b^n(\bar{\theta}_\epsilon^n)|V=1) \to 0$; or alternatively 2) $\liminf \Pr(P^n = b^n(\bar{\theta}_\epsilon^n)|V=1) > 0$ for all $\epsilon < \eta \lim \mathbb{E}[P^n|V=0]$ where $\bar{\theta}_\epsilon^n$ is the type defined in Eq. A.4.

Case 1: Our assumption that $\Pr(P^n = b^n(\bar{\theta}_\epsilon^n)|V=1) \to 0$ implies $\lim \Pr(P^n > b^n(\bar{\theta}_\epsilon^n)|V=1) = \lim \Pr\left(Y_s^n(k+1) > \bar{\theta}_\epsilon^n|V=1\right) = 1-\epsilon$. Therefore, $\lim u(b^n(\bar{\theta}_\epsilon^n)|\bar{\theta}_\epsilon^n) \leq \lim\left(\Pr(V=1|\bar{\theta}_\epsilon^n)\epsilon - \Pr(V=0|\bar{\theta}_\epsilon^n)\mathbb{E}[P^n|V=0]\right)$. However, $\lim \mathbb{E}[P^n|V=0] > 0$ implies that $\lim u\left(b^n(\bar{\theta}_\epsilon^n)|\bar{\theta}_\epsilon^n\right) < 0$ because $\epsilon < \eta \lim \mathbb{E}[P^n|V=0]$ and because $\frac{\Pr(V=0|\bar{\theta}_\epsilon^n)}{\Pr(V=1|\bar{\theta}_\epsilon^n)} = \frac{1}{l(\bar{\theta}_\epsilon^n)} > \eta$ leading to a contradiction. Therefore, $\lim \mathbb{E}[P^n|V=0] = 0$.

Case 2: Our assumption $\liminf \Pr(P^n = b^n(\bar{\theta}_\epsilon^n)|V=1) > 0$ implies $\lim \Pr(Y_s^n(k+1) \in \left\{\theta : b^n(\theta) = b^n(\bar{\theta}_\epsilon^n)\right\}|V=1) > 0$. In other words, $\bar{\theta}_\epsilon^n$ bids in a pool and $\lim \sqrt{n} F_s^n(\{\theta : b^n(\theta) = b^n(\bar{\theta}_\epsilon^n)\}|V=1) > 0$. However, such a pool is not possible if $\lim \Pr(P^n = b^n(\bar{\theta}_\epsilon^n)|V=1) > 0$ and $\lim \Pr(P^n = b^n(\bar{\theta}_\epsilon^n)|V=0) = 0$ by Lemma A.4.

Information aggregation and $\sqrt{n}(F_s^n(\theta_s^n(1)|0) - F_s^n(\theta_s^n(0)|0)) \to \infty$ together imply that $\lim \Pr(P^n \leq b^n(\theta_s^n(0))|V=1) = 0$. Therefore, $\lim u(b^n(\theta_s^n(0))|\theta_s^n(0)) = 0$. However,

$0 = \lim u(b^n(\theta_s^n(0))|\theta_s^n(0)) \geq \lim u(b=1|\theta_s^n(0)) = \lim \Pr(V=1|\theta_s^n(0))(1-\mathbb{E}[P^n|V=1])$, i.e., $\lim \mathbb{E}[P^n|V=1]=1$. □

□

The following lemma also provides conditions for information aggregation that we frequently use.

Lemma A.9. *Fix a sequence of bidding equilibria* $\mathbf{H}$*. If* $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$ *or if* $F_s(1|1) \geq F_s(1|0)$ *and* $F_s(1|1) > \kappa_s$*, then there is no pooling by pivotal types and price converges to value.*

Proof. For the following argument, note that $F_s(1|1) \geq F_s(1|0)$ and $F_s(1|1) > \kappa_s$ together imply that $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$ (see Lemma A.3). Under the lemma's assumptions the pivotal types are distinct and pooling by pivotal types is incompatible with equilibrium by Lemma A.3. However, then Lemma 2.2 implies that information is aggregated and Claim A.7 further implies that price converges to value because $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$. □

A.2. The Market Selection Lemmata. In this section, we characterize market selection. Throughout the section we use $a_s^H(\theta) := a^H(\theta)$ and $a_r^H(\theta) := 1 - a^H(\theta)$ to simplify exposition.

Lemma A.10 (Single Crossing Lemma)**.** *Suppose that* $a_m^H(\theta') > 0$ *for some type* θ' *in an equilibrium* H*. If* $u^H(m,b(\theta')|V=0) < u^H(m',b|V=0)$*, for* $m \neq m'$ *and some bid* $b \geq 0$*, then* $u(m,b(\theta')|\theta) > u(m',b|\theta)$ *for all* $\theta > \theta'$ *such that* $\theta \notin \mathcal{E}(\theta')$*.*

Proof. Fix an equilibrium H. For the remainder of the proof we suppress reference to the equilibrium H. Note that $u(m,b'|\theta,V=v) = u(m,b'|V=v)$ for any b', θ and v. Writing down the profit for type θ from bidding b in market m, we obtain $u(m,b|\theta) = u(m,b|V=0)\Pr(V=0|\theta) + u(m,b|V=1)\Pr(V=1|\theta)$. Our initial assumption that $a_m(\theta') > 0$ implies $u(m,b(\theta')|\theta') - u(m',b|\theta') \geq 0$. Moreover, $u(m,b(\theta')|\theta') - u(m',b|\theta') \geq 0$ and $u(m,b(\theta')|V=0) < u(m',b|V=0)$ together imply that $u(m,b(\theta')|V=1) - u(m',b|V=1) > 0$. Hence, if $\theta > \theta'$ and $\theta \notin \mathcal{E}(\theta')$, then

$$\begin{aligned}(u(m,b(\theta')|V=0)-u(m',b|V=0))\Pr(V=0|\theta)+(u(m,b(\theta')|V=1)-u(m',b|V=1))\Pr(V=1|\theta) >\\ (u(m,b(\theta')|V=0)-u(m',b|V=0))\Pr(V=0|\theta')+(u(m,b(\theta')|V=1)-u(m',b|V=1))\Pr(V=1|\theta')\\ = u(m,b(\theta')|\theta')-u(m',b|\theta') \geq 0\end{aligned}$$

because $\Pr(V=1|\theta) > \Pr(V=1|\theta')$. □

Below we define $\hat{\theta}_m$ for $m \in \{s,r\}$ as the smallest type which wins a good with positive probability if $V=0$ at the limit as n grows large, i.e., this type is the smallest "active" type in state $V=0$.

Definition A.1. Fix a sequence of symmetric distributional strategies $\{H^n\}$. If $F_m(1|0) \geq \kappa_m$, let $\theta_m^n(\epsilon) := \inf\{\theta : H^n([0,1] \times m \times (b^n(\theta),1]|0) < \kappa_m - \epsilon\}$, $\hat{\theta}_m(\epsilon) := \limsup \theta_m^n(\epsilon)$, and $\hat{\theta}_m := \inf_{\epsilon>0} \theta_m(\epsilon)$. If $F_m(1|0) < \kappa_m$, let $\hat{\theta}_m = \inf\{\theta : F_m(\theta|0) > 0\}$, and $\hat{\theta}_m = 1$ if the set is empty.

Suppose that $F_s(1|0) \geq \kappa_s$. The definition above selects type $\hat{\theta}_s = \theta_s(0)$ if the bidding function b^n is strictly increasing at $\theta_s^n(0)$ for sufficiently large n. The definition has more bite if, on the other hand, $\theta_s^n(0)$ submits a pooling bid. If $\theta_s^n(0)$ submits a pooling bid, then there are types $\underline{\theta}_p^n \leq \theta_s^n(0) \leq \theta_p^n$ who submit the same bid as $\theta_s^n(0)$. There are two cases to consider: In the first case $\theta_s(0) = \lim \theta_p^n$. Then the definition selects $\hat{\theta}_s = \theta_s(0)$. In the second case, if $\theta_s(0) < \lim \theta_p^n$, then the definition selects $\hat{\theta}_s = \lim \underline{\theta}_p^n$.

Lemma A.11. *Suppose that for an equilibrium sequence* $\mathbf{H}$ *we have that* $\lim \mathbb{E}(P_s^n|0) = 0$ *and* $\lim \mathbb{E}(P_r^n|0) > 0$, *then* $\lim a_r^n(\theta) = 1$ *for any* $\theta > \hat{\theta}_r$.

Proof. The fact that $\lim_n \mathbb{E}(P_s^n|V=0) = 0$ implies that $\lim u^n(s,b|V=0) = 0$ for any b. Pick an $\epsilon > 0$ and a sequence of types $\theta^n \in [\hat{\theta}_r^n(\epsilon/2), \hat{\theta}_r^n(\epsilon)]$ such that the limits $\lim \theta^n$, $\lim \hat{\theta}_r^n(\epsilon/2)$, $\lim \hat{\theta}_r^n(\epsilon)$ all exist and $a_r^n(\theta^n) > 0$. The probability that $P_r^n \leq b_r^n(\theta^n)$ converges to one in state 0. Therefore, the probability that θ^n wins an object in state 0 converges to one if this type does not bid in an atom along the sequence. Otherwise, the probability that this type wins is at least $\epsilon/2$ (see Lemma A.2 for this calculation). Hence, $\lim u(r, b_r^n(\theta^n)|V=0) \leq -\frac{\epsilon}{2} \lim \mathbb{E}[P_r^n|V=0] < 0 = \lim u^n(s,b|V=0)$. Lemma A.10 then implies that $\lim u(r, b_r^n(\theta^n)|\theta) > \lim u(s,b|\theta)$ for any b and any type $\theta > \lim \theta^n$ such that $\theta \notin \mathcal{E}(\lim \theta^n)$ and therefore $a_r(\theta) = 1$. Similarly, if $\theta > \lim \theta^n$ and $\theta \in \mathcal{E}(\lim \theta^n)$, then $a_r^n(\theta) = 1$. This is because we can pick, without loss of generality, a pure and increasing representation of the market selection strategy a_r^n over $\mathcal{E}(\lim \theta^n)$. Since ϵ is arbitrary and $\hat{\theta}_r = \inf_\epsilon \hat{\theta}_r(\epsilon)$ we conclude that $\lim a_r^n(\theta) = 1$ for any $\theta > \hat{\theta}_r$. □

Lemma A.12. *If* $\lim a_r^n(\theta) = 1$ *for all* $\theta > \theta_{en}$ *and* $\kappa_s > \bar{\kappa}$, *then either* $\theta_s(0) > \theta_s(1)$ *or* $\kappa_s > F_s(1|1)$. *Alternatively, if* $\lim a_r^n(\theta) = 0$ *for all* $\theta < \theta_{en}$ *and* $\kappa_s < \bar{\kappa}$, *then* $\theta_s(0) < \theta_s(1)$.

Proof. We argue that $\lim a_r^n(\theta) = 1$ for all $\theta > \theta_{en}$, $\kappa_s \leq F_s(1|1)$, and $\kappa_s > \bar{\kappa}$, together imply that $\theta_s(0) - \theta_s(1) > 0$. Let L_1 denote the set of measurable functions $\alpha : [0,1] \to [0,1]$ and consider the optimization problem

$$W(\kappa_s, \theta_{en}) = \max_{\alpha \in L_1} \frac{\int_0^{\theta_{en}} \alpha(\theta) dF(\theta|1)}{\int_0^{\theta_{en}} \alpha(\theta) dF(\theta|0)} \; s.t. \int_0^{\theta_{en}} \alpha(\theta) dF(\theta|1) = \kappa_s.$$

MLRP implies that $W(\kappa_s,\theta_{en})=\frac{F([\theta',\theta_{en}]|1)}{F([\theta',\theta_{en}]|0)}$ where θ' is the type such that $F([\theta',\theta_{en}]|1)=\kappa_s$.[20] If $\kappa_s>\bar{\kappa}_{en}=F([\theta^*(\theta_{en}),\theta_{en}]|0)=F([\theta^*(\theta_{en}),\theta_{en}]|1)$, then $\theta'<\theta^*(\theta')$, and MLRP implies $F([\theta',\theta^*(\theta_{en})]|0)>F([\theta',\theta^*(\theta_{en})]|1)$. Therefore, $W(\kappa_s,\theta_{en})<1$.

Assume $\kappa_s>F_s(1|1)$ and define $\alpha^*(\theta)$ as the function which is equal to zero for all $\theta\leq\theta_s(1)$ and equal to $a_s(\theta)$ for all $\theta>\theta_s(1)$. This function α^* is feasible for the above maximization problem. Therefore, we obtain $\frac{\bar{F}_s(\theta_s(1)|1)}{\bar{F}_s(\theta_s(1)|0)}=\frac{\int_{\theta_s(1)}^{\theta_{en}}a_s(\theta)dF(\theta|1)}{\int_{\theta_s(1)}^{\theta_{en}}a_s(\theta)dF(\theta|0)}=\frac{\int_0^{\theta_{en}}\alpha^*(\theta)dF(\theta|1)}{\int_0^{\theta_{en}}\alpha^*(\theta)dF(\theta|0)}\leq W(\kappa_s,\theta_{en})<1$. Hence, $\theta_s(0)>\theta_s(1)$.

We now argue that if $\lim a_r^n(\theta)=0$ for all $\theta<\theta_{en}$ and $\kappa_s<\bar{\kappa}$, then $\theta_s(0)<\theta_s(1)$. Define θ' as the type such that $F([\theta',\theta_{en}]|1)=\kappa_s$. Consider the following minimization problem $W(\kappa_s,\theta')=\min_{\alpha\in L_1}\frac{\int_{\theta'}^1\alpha(\theta)dF(\theta|1)}{\int_{\theta''}^1\alpha(\theta)dF(\theta|0)}$ $s.t. \int_{\theta'}^1\alpha(\theta)dF(\theta|1)=\kappa_s$. MLRP implies that $W(\kappa_s,\theta')=\frac{F([\theta',\theta_{en}]|1)}{F([\theta',\theta_{en}]|0)}$. Also, if $\kappa_s<\bar{\kappa}=F([\theta^*(\theta_{en}),\theta_{en}]|0)=F([\theta^*(\theta_{en}),\theta_{en}]|1)$, then $\theta'>\theta^*(\theta_{en})$, and hence $W(\kappa_s,\theta')>1$ by MLRP. Define $\alpha^*(\theta)$ as the function that is equal to zero for all $\theta\leq\theta_s(1)$ and equal to $a_s(\theta)$ for all $\theta>\theta_s(1)$. This α^* is feasible for the minimization problem. Therefore, $\frac{\bar{F}_s(\theta_s(1)|1)}{\bar{F}_s(\theta_s(1)|0)}=\frac{\int_{\theta_s(1)}^1 a_s(\theta)dF(\theta|1)}{\int_{\theta_s(1)}^1 a_s(\theta)dF(\theta|0)}=\frac{\int_{\theta'}^1\alpha^*(\theta)dF(\theta|1)}{\int_{\theta'}^1\alpha^*(\theta)dF(\theta|0)}\geq W(\kappa_s,\theta')>1$ and hence $\theta_s(1)>\theta_s(0)$. □

A.3. Proof of Theorem 3.1. In the following lemma we characterize behavior in market r. We then use this lemma to prove Theorem 3.1.

Lemma A.13. *If $c>0$, then $F_r(1|0)<F_r(1|1)\leq\kappa_r$ along any equilibrium sequence. Moreover, the price in market r converges to c almost surely if $V=0$ and converges to a random variable $P_r(1)$ if $V=1$. The random variable $P_r(1)$ is equal to c with probability $q>0$ and is equal to 1 with the remaining probability.*

Proof. The following three steps will together prove the result.

Step 1. $F_r(1|0)<F_r(1|1)\leq\kappa_r$. We will argue that $F_r(1|0)<F_r(1|1)$. If $F_r(1|0)<F_r(1|1)$, then we must have $F_r(1|1)\leq\kappa_r$. This is because $F_r(1|0)<F_r(1|1)$ and $F_r(1|1)>\kappa_r$ together imply that $P_r^n\to 1$ if $V=1$ by Lemma A.9. But this is not possible because all the bidders in market r would then earn negative profits.

We now show $F_r(1|0)<F_r(1|1)$. First, suppose that $F_r(1|0)>F_r(1|1)$. This implies that $F_s(1|0)<F_s(1|1)$. There are two cases: $F_s(1|1)>\kappa_s$ and $F_s(1|1)\leq\kappa_s$. If $F_s(1|1)>\kappa_s$, then $P_s^n\to 0$ if $V=0$ by Lemma A.9, and if $F_s(1|1)\leq\kappa_s$, then again $P_s^n\to 0$ if $V=0$ because $F_s(1|0)<\kappa_s$. However, if $P_s^n\to 0$ when $V=0$, then $a_r(\theta)=1$ for all $\theta>\hat{\theta}_r$ by Lemma A.11. However, if $F_r(1|0)<\kappa_r$, then $a_r(\theta)=1$ for all $\theta>\hat{\theta}_r$ implies that $F_r(1|1)>F_r(1|0)$, which contradicts our initial assumption. On the other

[20] In other words, the function $\alpha^*(\theta)$, which is equal to one if $\theta\geq\theta'$ and equal to zero otherwise is a maximizer of the problem.

hand, if $F_r(1|0)\geq\kappa_r$, then $F_r(1|1)>\kappa_r$ However, $F_r(1|1)>\kappa_r$ and $a_r(\theta)=1$ for all $\theta>\hat{\theta}_r$ together imply that $P_r^n\to 1$ if $v=1$ by Lemma A.9, which is not possible.

Second, suppose that $F_r(1|0)=F_r(1|1)$. There are two cases to consider: $F_r(1|1)>\kappa_r$ and $F_r(1|1)\leq\kappa_r$. If $F_r(1|1)>\kappa_r$, then $P_r^n\to 1$ if $V=1$ by Lemma A.9, which is not possible. Alternatively, If $F_r(1|1)\leq\kappa_r$, then $F_s(1|1)>\kappa_s$. However, $F_s(1|0)=F_s(1|1)$ and $F_s(1|1)>\kappa_s$ together imply by Lemma A.9 that $P_s^n\to 0$ if $V=0$. However, as argued previously, if $P_s^n\to 0$ if $V=0$ and if $F_r(1|0)\leq\kappa_r$, then almost all types in market r win an object when $V=0$ at a price which is at least c. Therefore, $a_r(\theta)=1$ for all $\theta>\hat{\theta}_r$ by Lemma A.11. Thus, we conclude that $F_r(1|1)>F_r(1|0)$ because $a_r(\theta)=1$ for all $\theta>\hat{\theta}_r$. However, this contradicts that $F_r(1|0)=F_r(1|1)$, as we initially assumed.

Step 2. Assume $\lim\sqrt{n}(F_r^n(1|1)-\kappa_r)>-\infty$ –i.e., there are more bidders than objects in market r with positive probability in state 1. We have $b_r^n(\theta)\to 1$ for any type θ that bids in market r.

For any $\epsilon>0$, pick θ^n such that $\Pr(Y_r^{n-1}(n\kappa_r)\in(0,\theta^n)|V=1)\leq\epsilon$, and recall that that $Y_r^{n-1}(n\kappa_r)=0$ if there are fewer than $n\kappa_r+1$ bidders in market r. For sufficiently small ϵ, $\lim\Pr(Y_r^{n-1}(n\kappa_r)\geq\theta^n|V=1)>0$ because $\lim\Pr(Y_r^{n-1}(n\kappa_r)=0|V=1)<1$ by assumption.

We argue that $\lim b_r^n(\theta)=1$ for any $\theta>\lim\theta^n$. Any type θ^n in this sequence can ensure winning an object by submitting a bid equal to one in the auction. Therefore, $u(r,b_r^n(\theta^n)|\theta^n)=\mathbb{E}[V-P_r^n|b_r^n(\theta^n)\ win,\theta]\Pr(b_r^n(\theta^n)\ win|\theta)\geq u(r,b=1|\theta^n)$. Noticing that, $u(r,b=1|\theta^n)=\mathbb{E}[V-P_r^n|b_r^n(\theta^n)\ win,\theta]\Pr(b_r^n(\theta^n)\ win|\theta)+\mathbb{E}[V-P_r^n|b_r^n(\theta^n)\ lose,\theta]\Pr(b_r^n(\theta^n)\ lose|\theta)$ we find $\mathbb{E}[V-P_r^n|b_r^n(\theta^n)\ lose,\theta]\Pr(b_r^n(\theta^n)\ lose|\theta)\leq 0$.

First, note that $\Pr(P_r^n\geq b_r^n(\theta^n)|V=0)\leq e^{-\frac{\delta^2 nF_r^n(1|0)}{2+\delta}}$ by applying Chernoff's inequality (see Janson et al. (2011, Theorem 2.1) where $\delta:=\frac{\kappa_r}{F_r^n(1|0)}-1$. Therefore, $\Pr(P_r^n\geq b_r^n(\theta^n)|V=0)\leq e^{-\frac{\delta^2 nF_r^n(1|0)}{2+\delta}}$. Suppose that $\lim\Pr(P_r^n=b_r^n(\theta^n)|V=1)=0$. Then $\Pr(b_r^n(\theta^n)\ lose|V=1)=\lim\Pr(P_r^n\geq b_r^n(\theta^n)|V=1)>0$. The fact that $\mathbb{E}[V-P_r^n|P_r^n\geq b_r^n(\theta^n),\theta]\Pr(P_r^n\geq b_r^n(\theta^n)|\theta)\leq 0$ implies that $\lim(1-\mathbb{E}[P_r^n|P_r^n\geq b_r^n(\theta^n),V=1])\leq c\lim\frac{\Pr(P_r^n\geq b_r^n(\theta^n)|V=0)}{\Pr(P_r^n\geq b_r^n(\theta^n)|V=1)l(\theta^n)}=c\lim\frac{e^{-\frac{\delta^2 nF_r^n(1|0)}{2+\delta}}}{\Pr(P_r^n\geq b_r^n(\theta^n)|V=1)l(\theta^n)}=0$, i.e., $\lim b_r^n(\theta)=1$ for almost all $\theta>\lim\theta^n$. Alternatively, suppose that $\lim\Pr(P_r^n=b_r^n(\theta^n)|V=1)>0$. If $\lim\Pr(P_r^n=b_r^n(\theta^n)|V=1)>0$, then Lemma A.2 implies that there is a constant A such that $\Pr(P_r^n=b_r^n(\theta^n),b_r^n(\theta^n)\ lose|V=1)\geq A/\sqrt{n}$ for all sufficiently large n. Therefore, $(1-b_r^n(\theta^n))\Pr(P_r^n=b_r^n(\theta^n),b_r^n(\theta^n)\ lose|V=1)l(\theta^n)-c\Pr(b_r^n(\theta^n)\ lose|V=0)\leq 0$, i.e., $\lim(1-b_r^n(\theta^n))\leq\lim\frac{c}{A}\sqrt{n}e^{-\frac{\delta^2 nF_r^n(1|0)}{2+\delta}}=0$. Therefore, $\lim b_r^n(\theta)=1$ for all $\theta\geq\lim\theta^n$.

Step 3 The price in market r converges to c almost surely if $V=0$ and converges to a random variable $P_r(1)$ if $V=1$. The random variable $P_r(1)$ is equal to c with probability $q>0$ and is equal to 1 with the remaining probability.

The fact that the price converges to c almost surely if $V=0$ follows from the LLN and the fact that $F_r(1|0) < \kappa_r$. Also, note that $\lim\sqrt{n}(F_r^n(1|1)-\kappa_r) < \infty$. This is because if $\lim\sqrt{n}(F_r^n(1|1)-\kappa_r) = \infty$, then the price clears at the bid of some type with probability one in state 1. However, the previous claim showed that $b_r^n(\theta)\to 1$ for all θ. But then $P_r(1)\to 1$, which implies that all bidders make a loss. The fact that $\lim\sqrt{n}(F_r^n(1|1)-\kappa_r)<\infty$ implies that $P_r(1)$ is equal to c with probability $q>0$. With the remainder of the probability, i.e., with probability $1-q$, the auction clears at the bid of some type θ and $b_r^n(\theta)\to 1$. Therefore, the auction price is equal to 1 with probability $1-q$. □

Proof of Theorem 3.1. Fix an equilibrium sequence **H**. If $c>0$, then information is not aggregated in market r by Lemma A.13. We now prove the other assertions in the theorem.

If $c>0$ and $\kappa_s>\bar{\kappa}$, then information is not aggregated in market s. Assume, on the way to a contradiction, that information is aggregated in market s. First suppose that $\lim_n\mathbb{E}[P_s^n]=0$. Note that $\lim_n\mathbb{E}[P_r^n|V=0]=c>0$ by Lemma A.13. Therefore, if $\lim_n\mathbb{E}[P_s^n]=0$, then all types would prefer to submit a bid equal to one in market s for all sufficiently large n. But if all types bid in auction s, then $F_s(\theta_s(1)|1)>F_s(\theta_s(0)|1)$ and Lemma A.9 implies that $\lim_n\mathbb{E}[P_s^n|V=0]=0$, $\lim_n\mathbb{E}[P_s^n|V=1]=1$ and therefore $\lim_n\mathbb{E}[P_s^n]=1/2$ which contradicts that $\lim_n\mathbb{E}[P_s^n]=0$. Hence, if information is aggregated in auction, then price converges to value by Lemma A.8.

The fact that price converges to value in auction s implies that $\lim u^n(s,b_s^n(\theta)|\theta)=0$ for all θ. We first argue that $\lim a_r^n(\theta)=1$ for all $\theta>\theta_{en}$. Recall that $\hat{\theta}_r$ is the smallest type that wins and object in state 0 in market r (definition A.1). If $\theta>\hat{\theta}_r$, then $\lim a_r^n(\theta)=1$ by Lemma A.11 because $\lim_n\mathbb{E}[P_r^n|V=0]>0$ and $\lim_n\mathbb{E}[P_s^n|V=0]=0$. Also, note that $\hat{\theta}_r\leq\theta_{en}$ because if $\hat{\theta}_r>\theta_{en}$, then $\lim_n\mathbb{E}[P_r^n|V=1]=c$ because $\theta_r^F(1)\leq\theta_{en}$ by Definition 3.1. However, if $\lim_n\mathbb{E}[P_r^n|V=1]=c$, then $\lim u^n(r,b_r^n(\theta)|\theta)>0$ for all $\theta\in(\theta_{en},\hat{\theta}_r)$, contradicting that $\hat{\theta}_r\leq\theta_{en}$.

If $F_s(1|1)<\kappa_s$, then $P_s^n\to 0$ in state 1 showing that information is not aggregated in market s. Instead suppose that $F_s(1|1)\geq\kappa_s$. Lemma A.12 shows that if $\lim a_r^n(\theta)=1$ for all $\theta>\theta_{en}$ and if $\kappa_s>\bar{\kappa}$, then $\theta_s^n(0)-\theta_s^n(1)\geq 0$ for all sufficiently large n. If $F_s(1|1)\geq\kappa_s$, then $\theta_s^n(0)-\theta_s^n(1)\geq 0$ for all sufficiently large n; however, this contradicts our initial assumption that information is aggregated in market s. This is because information aggregation in market s implies that $\theta_s^n(1)-\theta_s^n(0)>0$ for all sufficiently large n.

If $c>0$ and $\kappa_s<\bar{\kappa}$, then information is aggregated in market s. We prove this by looking at two cases. First, assume that $\theta_{en}=\inf\{\theta:\Pr(V=1|\theta)>c\}$. The fact that $\kappa_s<\bar{\kappa}_{en}$ implies that $F_s(\theta_s(1)|1)>F_s(\theta_s(0)|1)$, even if all $\theta\geq\theta_{en}$ choose market r

by Lemma A.12. If $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$, then Lemma A.9 implies that information is aggregated. Second, assume that $\theta_{en} = \theta_r^F(1)$. Lemma A.13 implies that $F_r(1|1) \leq \kappa_r$. However, if $F_r(1|1) \leq \kappa_r$, then Lemma A.12 implies that $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$. If $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$, then Lemma A.9 implies that information is aggregated. □

A.4. Proof of Proposition 4.1.

Proof. The fact that $u(r|0) = -c < 0$ and the fact that any type $\theta \in \mathcal{E}(0)$ can guarantee a payoff equal to zero by bidding zero in auction s implies that $a^n(\theta) = 1$ for $\theta \in \mathcal{E}(0)$, i.e., $F_s^n(\mathcal{E}(0)|0) = 1 - g$.

Step 1. For any selection function $a(\cdot)$, there is a unique bidding equilibrium where $b_m^n(\theta) = 0$ for each $\theta \in \mathcal{E}(0)$, $b_m^n(\theta) = 1$ for each $\theta \in \mathcal{E}(1)$, and $b_m^n(\theta) = \mathbb{E}[V|Y_m^{n-1}(k_m) = \theta]$ for each $\theta \in \mathcal{E}(1/2)$. An explicit formula for the bidding function is given by Eq. (4.1).

Without loss of generality we focus on market s. Any type $\theta \in \mathcal{E}(0)$ would never submit a bid that exceeds 0 because they are certain that the value of the object is equal to zero. Similarly, any type $\theta \in \mathcal{E}(1)$ always submits a bid equal to one because they are certain that the value of the object is equal to one.

We argue that if $F_s^n(\mathcal{E}(1/2)|1) > 0$ and $F_s^n(\mathcal{E}(1)|1) > 0$, then the bidding distribution has no atoms except at $b = 1$ and $b = 0$ and therefore $b_s^n(\theta) = \mathbb{E}[V|Y_s^{n-1}(k_s) = \theta, \theta_i = \theta] = \mathbb{E}[V|Y_s^{n-1}(k_s) = \theta]$ for each $\theta \in \mathcal{E}(1/2)$ by Lemma 2.1. To see that the bid distribution is atomless, define an auxiliary type distribution G with three distinct signals with $G(\mathcal{E}(1/2)|v) = F_s(\mathcal{E}(1/2)|v)$, $G(\mathcal{E}(1)|v) = F_s(\mathcal{E}(1)|v)$ and $G(\mathcal{E}(0)|v) = 1 - G(\mathcal{E}(1/2)|v) - G(\mathcal{E}(1)|v)$. In this auxiliary type distribution all types that choose market r are assumed to receive a signal $\theta \in \mathcal{E}(0)$. Given this auxiliary type distribution no type $\theta \in \mathcal{E}(1/2)$ would bid in an atom. This is because the auxiliary type distribution satisfies MLRP and the uninformed bidding in an atom would contradict Lemma 7 in Pesendorfer and Swinkels (1997) which rules out such atoms.

We now argue if $F_s^n(\mathcal{E}(1)|1) = 0$, then $b^n(\theta) = 1/2 = \mathbb{E}[V|Y_s^{n-1}(k_s) = \theta]$ for each $\theta \in \mathcal{E}(1/2)$. Note that any type $\theta \in \mathcal{E}(1/2)$ would always under cut any atom $b > 1/2$ and out bid any atom $b < 1/2$. Therefore, types $\theta \in \mathcal{E}(1/2)$ can bid in an atom only at $b = 1/2$. If the bid distribution is strictly increasing over some interval of types, then Lemma 2.1 implies that $b^n(\theta) = \mathbb{E}[V|Y_s^{n-1}(k_s) = \theta] = \mathbb{E}[V] = 1/2$. Therefore, we conclude that all types $\theta \in \mathcal{E}(1/2)$ submit a bid equal to 1/2, i.e., $b^n(\theta) = 1/2 = \mathbb{E}[V|Y_s^{n-1}(k_s) = \theta]$ for each $\theta \in \mathcal{E}(1/2)$.

Step 2. There exists $\theta_1 \in \mathcal{E}(1/2)$ and $\theta_2 \in \mathcal{E}(1)$ such that all types $\theta \in [0, \theta_1) \cup (2/3, \theta_2]$ selecting market s, all others selecting market r, and each type submitting a bid equal to $b_m^n(\theta)$ is an equilibrium. The proof, which uses Kakutani's fixed point theorem, is given in the online appendix.

Step 3. Along any equilibrium sequence $u^n(\theta)\to 0$ for any $\theta\in\mathcal{E}(1/2)$.

Note that either $\lim F_s^n(\mathcal{E}(1)|1)+F_s^n(\mathcal{E}(1/2)|1)>\kappa_s$ or $\lim F_r^n(\mathcal{E}(1)|1)+F_r^n(\mathcal{E}(1/2)|1)>\kappa_r$ along any subsequence where these limits exist. Suppose, without loss of generality, $\lim F_s^n(\mathcal{E}(1)|1)+F_s^n(\mathcal{E}(1/2)|1)>\kappa_s$. If $F_s^n(\mathcal{E}(1)|1)>0$ for all sufficiently large n, then bidding in market s is increasing in $\theta\in\mathcal{E}(1/2)$. Therefore, there exists a $\theta'\in\mathcal{E}(1/2)$ that bids in market s and wins an object in state $V=1$ with probability converging to zero. This type's equilibrium payoff $u^n(\theta')$ converges to zero. But since all types in $\theta\in\mathcal{E}(1/2)$ have identical information, $u^n(\theta)=u^n(\theta')\to 0$ for any $\theta\in\mathcal{E}(1/2)$. If, on the other hand, $F_s^n(\mathcal{E}(1)|1)=0$ for all sufficiently large n, then all types $\theta\in\mathcal{E}(1/2)$ submit a bid equal to $1/2$ and therefore we again find $u^n(\theta)\to 0$ for any $\theta\in\mathcal{E}(1/2)$.

Step 4. Suppose $\lim\frac{\sqrt{n}}{\sigma}\bar{F}_m^n(\mathcal{E}(1)|V=1)=x\in[0,\infty]$. Then $\lim b_m^n(\theta^n)=\frac{\phi(y)}{\phi(y+x)}/(1+\frac{\phi(y)}{\phi(y+x)})$ for any sequence such that $\{\theta^n\}\subset\mathcal{E}(1/2)$ and $\lim\frac{\sqrt{n}}{\sigma}(\kappa_m-\bar{F}_m^n(\theta^n|V=1))=y$ where $\sigma=\sqrt{\kappa_m(1-\kappa_m)}$ if such a sequence exists. Moreover, $\lim\Pr(P_m^n\le b_m^n(\theta^n)|V=1)=\Phi(y)$ and $\lim\Pr(P_m^n\le b_m^n(\theta^n)|V=0)=\Phi(y+x)$.

Any type θ^n's bid is given by $b_m^n(\theta^n)=\frac{h\left(Y_m^{n-1}(k_m)=\theta^n|V=1\right)}{h\left(Y_m^{n-1}(k_m)=\theta^n|V=0\right)}/(1+\frac{h\left(Y_m^{n-1}(k_m)=\theta^n|V=1\right)}{h\left(Y_m^{n-1}(k_m)=\theta^n|V=0\right)})$ where $h(Y_m^{n-1}(k_m)=\theta^n|V=1):=\frac{d}{d\theta}\Pr(Y_m^{n-1}(k_m)\le\theta^n|V=1)$, i.e., h is a binomial density. If $x\in[0,\infty)$, then Lemma A.6 implies that $b_m^n(\theta^n)\to\frac{\phi(y)}{\phi(y+x)}/(1+\frac{\phi(y)}{\phi(y+x)})=e^{yx+\frac{x^2}{2}}/(1+e^{yx+\frac{x^2}{2}})\in(0,1)$. Moreover, the central limit theorem implies that $\Pr(P_m^n\le b_m^n(\theta^n)|V=1)=\Phi(y)$ and $\Pr(P_m^n\le b_m^n(\theta^n)|V=0)=\Phi(y+x)$. If $x=\infty$, then $b_m^n(\theta^n)\to 1$ because $\frac{\sqrt{n}h\left(Y_m^{n-1}(k_m)=\theta^n|V=1\right)}{\sqrt{n}h\left(Y_m^{n-1}(k_m)=\theta^n|V=0\right)}\to\frac{\phi(y)}{\sqrt{n}h\left(Y_m^{n-1}(k_m)=\theta^n|V=0\right)}=\infty$ since $\sqrt{n}h(Y_m^{n-1}(k_m)=\theta^n|V=0)\to 0$.[21]

Step 5. Assume $c\ge 1/2$. In any equilibrium $F_r^n(\mathcal{E}(1/2)|1)=0$ and $F_s^n(\mathcal{E}(1)|1)>0$. Further assume $1-g<\kappa_r$. Then, along any equilibrium sequence, we have $P_r^n\to c$ in both states and $\lim\sqrt{n}F_s^n(\mathcal{E}(1)|1)<\infty$.

First we show $F_s^n(\mathcal{E}(1)|1)>0$. Suppose not, i.e., $F_s^n(\mathcal{E}(1)|1)=0$. Then $b_s^n(\theta)=1/2$ for any $\theta\in\mathcal{E}(1/2)$. If $c\ge 1/2$, any type $\theta\in\mathcal{E}(1)$ can get an object from auction s with probability one at an expected price strictly less than $1/2$. The expected price is less than $1/2$ because any $\theta\in\mathcal{E}(1/2)$ bids $1/2$ in market s and because the probability that the number of objects exceeds the number of bidders in the auction is positive. Therefore, if $F_s^n(\mathcal{E}(1)|1)=0$, then the payoff from participating in auction s is strictly greater than bidding in market r for $\theta\in\mathcal{E}(1)$ and this contradicts $F_s^n(\mathcal{E}(1)|1)=0$.

Second we show $F_r^n(\mathcal{E}(1/2)|1)=0$. Suppose not, i.e., $F_r^n(\mathcal{E}(1/2)|1)>0$. Any type $\theta\in\mathcal{E}(1/2)$ obtains a strictly positive payoff in auction s. This is because $u(s,b^n(\theta)|\theta)\ge u(s,b=0|\theta)$ for any $\theta\in\mathcal{E}(1/2)$. Moreover, $u(s,b=0|\theta)>0$ because the number of bidders in auction s is less than the number of objects with positive

[21] Note that $\lim\sqrt{n}h(Y_m^{n-1}(k_m)=\theta^n|V=0)\le\phi(y+x)$ for any finite x and $\lim_{x\to\infty}\phi(y+x)=0$.

probability because $F_r^n(\mathcal{E}(1/2)|1)>0$ and therefore a type submitting a bid equal to zero obtains an object with positive probability at a price equal to zero in state $V=1$. However, $\mathbb{E}[u(r,b|\theta)]\leq 1/2-c\leq 0$ for any $\theta\in\mathcal{E}(1/2)$ but this contradicts any type $\theta\in\mathcal{E}(1/2)$ choosing r over market s.

Finally, we show that if $1-g<\kappa_r$, then $\lim\sqrt{n}F_s^n(\mathcal{E}(1)|1)=0$. If $1-g<\kappa_r$, then $P_r^n\to c$ in both states because $\lim F_r^n(\mathcal{E}(1)|1)\leq 1-g<\kappa_r$ and $F_r^n(\mathcal{E}(1/2)|1)=0$. Assume to the contrary that $\lim\sqrt{n}F_s^n(\mathcal{E}(1)|1)=\infty$. The fact that $\lim F_s^n(\mathcal{E}(1/2)|1)=g>\kappa_s$, $\lim\sqrt{n}F_s^n(\mathcal{E}(1)|1)=\infty$, and Step 4 together imply that the price in market s converges to one in state $V=1$. However, then no type $\theta\in\mathcal{E}(1)$ would choose market s for sufficiently large n because the price in market r converges to c. But this contradicts $\lim\sqrt{n}F_s^n(\mathcal{E}(1)|1)=\infty$.

Step 6. Assume $c<1/2$. Along any equilibrium sequence, $F_s^n(\mathcal{E}(1)|1)>0$ for all sufficiently large n. Further assume $1-g<\kappa_r$. Then, along any equilibrium sequence, $\lim\sqrt{n}F_s^n(\mathcal{E}(1)|1)<\infty$, $\lim F_r^n(\mathcal{E}(1/2)|1)=\kappa_r+g-1$, $P_r^n\to c$ in state $V=0$, and $\lim\mathbb{E}[P_r^n|V=1]=1-c$.

We first show if $1-g<\kappa_r$ and $\lim F_s^n(\mathcal{E}(1)|1)=0$, then $\lim F_r^n(\mathcal{E}(1/2)|1)=\kappa_r+g-1$, $P_r^n\to c$ in state $V=0$, and $\lim\mathbb{E}[P_r^n|V=1]=1-c$. Subsequently, we will further establish $\lim\sqrt{n}F_s^n(\mathcal{E}(1)|1)=0$ if $1-g<\kappa_r$.

First, suppose that $\lim F_r^n(\mathcal{E}(1/2)|1)<\kappa_r+g-1$ along some subsequence. Then the price in market r converges to c in both states and each $\theta\in\mathcal{E}(1/2)$ that bids in market r wins an object with probability one at a price converging to c. However, then $\lim u^n(\theta)=1/2-c>0$ for any such type and this contradicts $u^n(\theta)\to 0$ for any $\theta\in\mathcal{E}(1/2)$ (Step 3).

Suppose instead that $\lim F_r^n(\mathcal{E}(1/2)|1)>\kappa_r+g-1$ along some subsequence. Then the price in market r converges to one in state $V=1$ by Step 4. But this would imply that the profit of any $\theta\in\mathcal{E}(1/2)$ that bids in market r is negative again leading to a contradiction.

Our finding that $\lim F_r^n(\mathcal{E}(1/2)|1)=\kappa_r+g-1$ further implies that the price in market r converges to c in state $V=0$ because $\lim F_r^n(\mathcal{E}(1/2)|0)<\kappa_r$ and $F_r^n(\mathcal{E}(1)|0)=0$. Moreover, $\lim\mathbb{E}[P_r^n|V=1]=1-c$. This is because if $\lim\mathbb{E}[P_r^n|V=1]<1-c$, then an uninformed type can make strictly positive profits by bidding one in market r leading to a contradiction. Similarly, if $\lim\mathbb{E}[P_r^n|V=1]>1-c$, then an uninformed type that wins an object with probability one in market r would make a loss again leading to a contradiction.

We next show $F_s^n(\mathcal{E}(1)|1)>0$ for all sufficiently large n. On the way to a contradiction assume that $F_s^n(\mathcal{E}(1)|1)=0$ for all sufficiently large n along some subsequence. This implies that $\lim\mathbb{E}[P_s^n|V=1]\geq\lim\mathbb{E}[P_r^n|V=1]$ because otherwise any type $\theta\in\mathcal{E}(1)$ would prefer to bid in market s instead of r for all sufficiently large

n. If $1-g>\kappa_r$ and $F_s^n(\mathcal{E}(1)|1)=0$, then $P_r\to 1$ and $P_s\to 1/2$. But this contradicts $\lim\mathbb{E}[P_s^n|V=1]\geq\lim\mathbb{E}[P_r^n|V=1]$. On the other hand if $1-g<\kappa_r$ and $F_s^n(\mathcal{E}(1)|1)=0$ for all sufficiently large n, then $b_s^n(\theta)=1/2$ for $\theta\in\mathcal{E}(1/2)$ for all sufficiently large n. Hence, $\lim\mathbb{E}[P_s^n|V=1]\leq 1/2<\lim\mathbb{E}[P_r^n|V=1]=1-c$ again leading to a contradiction.

We show that if $1-g<\kappa_r$, then $\lim F_s^n(\mathcal{E}(1)|1)=0$ along any equilibrium sequence. Assume $\lim F_s^n(\mathcal{E}(1)|1)>0$ along some subsequence. In this case $P_s^n\to 0$ in state $V=0$. This is because if $F_s^n(\mathcal{E}(1)|1)+F_s^n(\mathcal{E}(1/2)|1)>\kappa_s$, then price converges to one in state $V=1$ by Step 4. However, $\lim\mathbb{E}[P_r^n|V=1]\leq 1-c<1$, which implies that no $\theta\in\mathcal{E}(1)$ would bid in market s for sufficiently large n, contradicting $\lim F_s^n(\mathcal{E}(1)|1)>0$.[22] On the other hand, if $F_s^n(\mathcal{E}(1)|1)+F_s^n(\mathcal{E}(1/2)|1)\leq\kappa_s$, then $F_s^n(\mathcal{E}(1)|0)+F_s^n(\mathcal{E}(1/2)|0)=F_s^n(\mathcal{E}(1/2)|1)<\kappa_s$. But $F_s^n(\mathcal{E}(1)|0)+F_s^n(\mathcal{E}(1/2)|0)<\kappa_s$ implies that $P_s^n\to 0$ in state $V=0$. The fact that $\theta\in\mathcal{E}(1)$ bids in market s implies that $\lim\mathbb{E}[P_s^n|V=1]\leq\lim\mathbb{E}[P_r^n|V=1]\leq 1-c$. But then bidding one in market s gives any $\theta\in\mathcal{E}(1/2)$ a profit that is at least $c/2>0$ for large n because $P_s^n\to 0$ and $\lim\mathbb{E}[P_s^n|V=1]\leq 1-c$. We now further show that $\lim\sqrt{n}F_s^n(\mathcal{E}(1)|1)<\infty$. If $\lim\sqrt{n}F_s^n(\mathcal{E}(1)|1)=\infty$, then price in market s converges to one by Step 4 because $F_s^n(\mathcal{E}(1/2)|1)>\kappa_s$. However, then no type $\theta\in\mathcal{E}(1)$ would choose market s for sufficiently large n and this contradicts $F_s^n(\mathcal{E}(1)|1)>0$ for all sufficiently large n.

Step 7. Suppose that $1-g<\kappa_r$. The price P_s^n converges in distribution to a random variable P_s. The distribution functions of $\Pr(P\leq p|V=1)$ and $\Pr(P\leq p|V=0)$ are both atomless and strictly increasing on the interval [0,1]. Moreover, if $c>1/2$, then $\lim\mathbb{E}[P_s^n|V=1]=c$ and $\lim\mathbb{E}[P_s^n|V=0]=1-c$; if $c<1/2$, then $\lim\mathbb{E}[P_s^n|V=1]=1-c$ and $\lim\mathbb{E}[P_s^n|V=0]=c$; and if $c=1/2$, then the price converges in probability to 1/2 in both states.

If $1-g<\kappa_r$, then $F_s^n(\mathcal{E}(1/2)|1)>\kappa_s$ (see Steps 5 and 6). Suppose $\frac{\sqrt{n}}{\sigma}F_s^n(\mathcal{E}(1)|1)=x\in(0,\infty)$ where $\sigma=\sqrt{\kappa_s(1-\kappa_s)}$. For any finite y, pick a type $\theta^n\in\mathcal{E}(1/2)$ such that $\frac{\sqrt{n}}{\sigma}\left(\kappa_s-\bar{F}_s^n(\theta^n|V=1)\right)=y$. Such a type exists for all sufficiently large n because $F_s^n(\mathcal{E}(1/2)|1)>\kappa_s$. This type's bid is given by $b_s^n(\theta^n)\to\frac{\phi(y)}{\phi(y+x)}/(1+\frac{\phi(y)}{\phi(y+x)})=e^{yx+\frac{x^2}{2}}/(1+e^{yx+\frac{x^2}{2}})\in(0,1)$ by Step 4. Moreover, $\lim\Pr(P\leq p=b^n(\theta^n)|V=0)\to\Phi(y+x)$ and $\lim\Pr(P\leq p=b^n(\theta^n)|V=1)\to\Phi(y)$. Solving for y as a function of p using $p=e^{yx+\frac{x^2}{2}}/(1+e^{yx+\frac{x^2}{2}})$ we find $y=\frac{1}{x}\left(\ln\frac{p}{1-p}-\frac{x^2}{2}\right)$. So, $\lim\Pr(P\leq p|V=1)=\Phi((\ln\frac{p}{1-p}-x^2/2)/x)$ and $\lim\Pr(P\leq p|V=0)=\Phi((\ln\frac{p}{1-p}+x^2/2)/x)$. This formula expresses the limit price distribution in closed form and shows that the distribution is atomless and strictly increasing on [0,1]. Specifically, $\ln\frac{P}{1-P}$ has a normal distribution with mean $-x^2/2$ (or $x^2/2$) and standard deviation x in state $V=0$ (in state $V=1$) .

[22] If $1-g<\kappa_r$, then $\lim\mathbb{E}[P_r^n|V=1]\leq 1-c$ because otherwise no uninformed would bid in market r. However, if no uninformed bids in market r, then $\lim\mathbb{E}[P_r^n|V=1]=c<1-c$.

The facts that $\lim F_s^n(\mathcal{E}(1/2)|1) > \kappa_s$, $\lim F_s^n(\mathcal{E}(1)|1) = 0$, and increasing bids together imply that there is a type $\theta' \in \mathcal{E}(1/2)$ that wins an object with probability one in both states. Therefore, $\lim u^n(s, b_s^n(\theta')|\theta') = (1 - \lim \mathbb{E}[P_s^n|V=1] - \lim \mathbb{E}[P_s^n|V=0])/2 = 0$ and hence $1 - \lim \mathbb{E}[P_s^n|V=1] = \lim \mathbb{E}[P_s^n|V=0]$.

Suppose $c \geq 1/2$. In order for types $\theta \in \mathcal{E}(1)$ to be indifferent between the two markets, we must have $\lim \mathbb{E}[P_s^n|V=1] = c$ which in turn implies that $\lim \mathbb{E}[P_s^n|V=0] = 1-c$.

Suppose $c < 1/2$. If $c < 1/2$, then P_r^n converges to c. In order for types $\theta \in \mathcal{E}(1)$ to remain indifferent between the two markets, we must have $\lim \mathbb{E}[P_s^n|V=1] = \lim \mathbb{E}[P_r^n|V=1]$. Moreover, in order for types $\theta \in \mathcal{E}(1/2)$ to be indifferent between the two markets, we must have $\lim \mathbb{E}[P_s^n|V=0] = \lim \mathbb{E}[P_r^n|V=0]$ because $\lim \mathbb{E}[P_s^n|V=1] = \lim \mathbb{E}[P_r^n|V=1]$.

We now show that there is a unique value of x such that $\mathbb{E}[P_s|V=0;x] := \int_0^1 p d\Phi((\ln\frac{p}{1-p} + x^2/2)/x) = 1-c$ if $c \geq 1/2$ and $\mathbb{E}[P_s|V=0;x] = c$ if $c < 1/2$. To see that this equation has a unique solution set $P_s = Z/(1+Z)$. The variable Z has a lognormal distribution on support $[0,\infty)$ with parameters $-x^2/2$ and x. Theorem 5 in Levy (1973), which provides a sufficient condition for ordering lognormal distributions, implies that the distribution of Z is decreasing in x in the second order stochastic dominance order.[23] Moreover, $Z/(1+Z)$ is an increasing, concave function for $Z \geq 0$. Therefore, $\mathbb{E}[P|V=0;x]$ is decreasing in x and converges to zero as $x \to \infty$. Moreover, $\mathbb{E}[P|V=0;x]$ is equal to $1/2$ for $x=0$, which we argue below. Therefore, $\lim \mathbb{E}[P_s^n|V=0;x] = 1-c$ has a unique solution for any $c \geq 1/2$. The unique solution has $x=0$ if $c=1/2$ and $x>0$ if $c>1/2$. Also, $\lim \mathbb{E}[P_s^n|V=0;x] = c$ has a unique solution $x>0$ for any $c<1/2$.

We now complete the proof by showing that if $x=0$, then the price converges in probability to $1/2$. If $x=0$, then for any $y \in (-\infty,\infty)$ and sequence of $\{\theta^n\}$ such that $\sqrt{n}(\bar{F}_s^n(\theta^n|V=1) - \kappa_s)/\sigma = y$ we have $b^n(\theta^n) \to \frac{\phi(y)/\phi(y)}{1+\phi(y)/\phi(y)} = 1/2$. Moreover, $\lim \Pr(P \leq p = b^n(\theta^n)|V=1) = \lim \Pr(P \leq p = b^n(\theta^n)|V=0) \to \Phi(y)$. Therefore, $\lim \Pr(P < p = 1/2|V=v) = 0$ and $\lim \Pr(P > 1/2|V=v) = 0$ for $v = 0,1$, i.e., the price converges in probability to $1/2$.

Step 8. Assume $1-g > \kappa_r$. The price $P_s^n \to V$ and $P_r^n \to V + c(1-V)$ in probability along any equilibrium sequence.

We first argue $\lim \mathbb{E}[P_s^n|V=1] = \lim \mathbb{E}[P_r^n|V=1] = 1$ implies that $\lim \mathbb{E}[P_s^n|V=0] = 0$ and $\lim \mathbb{E}[P_r^n|V=0] = c$. If $\lim \mathbb{E}[P_r^n|V=1] = 1$, then $\lim F_r^n(\mathcal{E}(1/2)|1) = 0$ and therefore $\lim \mathbb{E}[P_r^n|V=0] = c$. This is because otherwise (i.e., if $\lim F_r^n(\mathcal{E}(1/2)|1) > 0$) any type

[23] Suppose $\ln Z_A$ is distributed $N(\mu_A, \sigma_A)$ and $\ln Z_B$ is distributed $N(\mu_B, \sigma_B)$. Theorem 5 in Levy (1973) proves that Z_A second order stochastically dominates Z_B if $\mu_A > \mu_B$, $\sigma_A < \sigma_B$, and $\mu_A + \sigma_A^2/2 \geq \mu_B + \sigma_B^2/2$. Substituting x for the standard deviation and -$x^2/2$ for the mean of the normal distribution then establishes our claim.

$\theta \in \mathcal{E}(1/2)$ that chooses market r would receive a payoff converging to $-c/2 < 0$. Moreover, $\lim F_r^n(\mathcal{E}(1/2)|1) = 0$ implies that there is a type $\theta' \in \mathcal{E}(1/2)$ that wins an object in state $V=0$ in market s with probability one because $b_s^n(\theta) = 0$ for $\theta \in \mathcal{E}(0)$ and $b_s^n(\theta) > 0$ for $\theta \in \mathcal{E}(1/2)$ (Step 1). However, then $\mathbb{E}[P_s^n|V=0] = 0$ because otherwise type $\theta' \in \mathcal{E}(1/2)$ would receive a negative payoff since $\lim \mathbb{E}[P_s^n|V=1] = 1$.

We will now complete the argument by showing that $\lim \mathbb{E}[P_s^n|V=1] = \lim \mathbb{E}[P_r^n|V=1] = 1$. Steps 5 and 6 establish that $F_s^n(\mathcal{E}(1)|1) > 0$ for all sufficiently large n and thus $\lim \mathbb{E}[P_s^n|V=1] \leq \lim \mathbb{E}[P_r^n|V=1]$ along any convergent sequence. Moreover, if $\lim F_s^n(\mathcal{E}(1)|1) = 0$, then $F_r^n(\mathcal{E}(1)|1) > 0$ for all sufficiently large n and hence $\lim \mathbb{E}[P_s^n|V=1] = \lim \mathbb{E}[P_r^n|V=1]$ along any convergent sequence.

Suppose $\lim F_s^n(\mathcal{E}(1)|1) = 0$. If $\lim F_s^n(\mathcal{E}(1)|1) = 0$, then $\lim F_r^n(\mathcal{E}(1)|1) = 1 - g > \kappa_r$. Since $b^n(\theta) = 1$ for all $\theta \in \mathcal{E}(1)$ and all n, we find $\lim \mathbb{E}[P_r^n|V=1] = 1$. Therefore, if $\lim F_s^n(\mathcal{E}(1)|1) = 0$, then $\lim \mathbb{E}[P_s^n|V=1] = \lim \mathbb{E}[P_r^n|V=1] = 1$.

Alternatively suppose $\lim F_s^n(\mathcal{E}(1)|1) > 0$. If $\lim(F_s^n(\mathcal{E}(1)|1) + F_s^n(\mathcal{E}(1/2)|1)) > \kappa_s$, then $\lim \mathbb{E}[P_s^n|V=1] = 1$ by Step 4 and $\lim \mathbb{E}[P_r^n|V=1] = 1$ because $1 = \lim \mathbb{E}[P_s^n|V=1] \leq \lim \mathbb{E}[P_r^n|V=1] \leq 1$. If $\lim(F_s^n(\mathcal{E}(1)|1) + F_s^n(\mathcal{E}(1/2)|1)) \leq \kappa_s$, then $\lim \mathbb{E}[P_s^n|V=0] = 0$ because $\lim F_s^n(\mathcal{E}(1/2)|0) = \lim F_s^n(\mathcal{E}(1/2)|1)) < \kappa_s$ and because $b_s^n(\theta) = 0$ for $\theta \in \mathcal{E}(0)$. However, if $\lim \mathbb{E}[P_s^n|V=0] = 0$, then $\lim \mathbb{E}[P_s^n|V=1] = 1$ and consequently $\lim \mathbb{E}[P_r^n|V=1] = 1$. This is because otherwise (i.e., if $\lim \mathbb{E}[P_s^n|V=1] < 1$) any type $\theta' \in \mathcal{E}(1/2)$ could make a positive profit in market s by submitting a bid equal to 1 contradicting $u(\theta') \to 0$ (Step 3). □

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B. Online Appendix

B.1. Proofs of Pooling Calculations. Given a pooling bid b_p^n, let the random variables L^n, G^n, and $X^n = L^n + G^n$ denote the number of losers, number of winners (or the number of objects left for the bidders that submit a bid equal to b_p^n), and number of bidders that submit a bid equal to b_p^n, respectively. Let $\bar{L}^n = \mathbb{E}[L^n|P^n = b_p^n], v$, $\bar{G}^n = \mathbb{E}[G^n|P^n = b_p^n, v]$ and $\bar{X}^n = \bar{L}^n + \bar{G}^n$. Given these definitions, $\Pr[b_p^n\, lose|P^n = b_p^n, v] = \mathbb{E}[L^n/X^n|P^n = b_p^n, v]$ and $\Pr[b_p^n\, win|P^n = b_p^n, v] = \mathbb{E}[G^n/X^n|P^n = b_p^n, v]$. For any type θ that submits the pooling bid, $\Pr(L^n = i|Y_s^n(k_s+1) = \theta, v) = bi(i; n-1-k_s, \frac{F_s^n([\underline{\theta}_p^n, \theta]|v)}{1-\bar{F}_s^n(\theta|v)})$ and $\Pr(X^n = i|Y_s^n(k_s+1) = \theta, v) = bi(i; n-1-k_s, \frac{F_s^n([\theta, \theta_p^n]|v)}{\bar{F}_s^n(\theta|v)})$. Therefore, $\mathbb{E}[L^n|Y_s^n(k_s+1) = \theta, v] = n\frac{F_s^n([\underline{\theta}_p^n, \theta]|v)}{1-\bar{F}_s^n(\theta|v)}\left(1-\kappa_s-\frac{1}{n}\right)$, $\mathbb{E}[X^n|Y_s^n(k_s+1) = \theta, v] = n\kappa_s\frac{F_s^n([\theta, \theta_p^n]|v)}{\bar{F}_s^n(\theta|v)}$, $\bar{L}^n = \int_{\underline{\theta}_p^n}^{\theta_p^n} n\frac{F_s^n([\underline{\theta}_p^n, \theta]|v)}{1-\bar{F}_s^n(\theta|v)}(1-\kappa_s-1/n)\Pr(Y_s^n(k_s+1) = \theta|v)d\theta$ and $\bar{X}^n = \int_{\underline{\theta}_p^n}^{\theta_p^n} n\kappa_s\frac{F_s^n([\theta, \theta_p^n]|v)}{\bar{F}_s^n(\theta|v)}\Pr(Y_s^n(k_s+1) = \theta|v)d\theta$.

We prove a somewhat stronger version of Lemma A.2 in Lemma B.1 below.

Lemma B.1. *If* $\lim \Pr(P^n \geq b_p^n|v=0)$*, then*

$$\lim \Pr(b_p^n\, lose|P^n = b_p^n, V = v) / \frac{\overline{F}_s^n(\underline{\theta}_p^n|v)(1-\bar{F}_s(\underline{\theta}_p^n|v))}{nF_s^n([\underline{\theta}_p^n, \theta_p^n]|v)(\kappa_s - \bar{F}_s^n(\underline{\theta}_p^n|v))} = 1.$$

Suppose $\lim \Pr(P^n = b_p^n|v) > 0$.

i. If $\lim F_s^n([\underline{\theta}_p^n, \theta_p^n]|v) > 0$*, then* $\lim \Pr(b_p^n\, win|P^n = b^n, v) = \lim \frac{\kappa_s - \bar{F}_s^n(\theta_p^n|v)}{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)}$.

ii. If $\sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\underline{\theta}_p^n|v)| \to \infty$*, then* $\lim \frac{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)}{F_s^n([\underline{\theta}_p^n, \theta^n(v)]|v)}\Pr(b_p^n\, lose|P^n = b_p^n, v) \in (0, \infty)$*;*

iii. If $\sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\underline{\theta}_p^n|v)| < \infty$*, then* $\lim \sqrt{n}F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)\Pr(b_p^n\, lose|P^n = b_p^n, v) \in (0, \infty)$;

iv. If $\sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\theta_p^n|v)| \to \infty$*, then* $\lim \frac{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)}{F_s^n([\theta^n(v), \theta_p^n]|v)}\Pr(b_p^n\, win|P^n = b_p^n, v) \in (0, \infty)$;

v. If $\sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\theta_p^n|v)| < \infty$*, then* $\lim \sqrt{n}F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)\Pr(b_p^n\, win|P^n = b_p^n, v) \in (0, \infty)$.

Proof of item i in Lemma B.1. Suppose that $Y_s^n(k_s+1) = \theta^n$, $\theta^n \in [\underline{\theta}_p^n, \theta_p^n]$ and $\bar{F}_s^n(\theta^n|v) \in (\kappa_s - \epsilon_1, \kappa_s + \epsilon_1)$. There are k_s bidders with signals above θ^n and the distribution of G^n is binomial, hence $\bar{G}_n = \frac{k_s\left(\bar{F}^n(\theta|v) - \bar{F}_s^n(\theta_p^n|v)\right)}{\bar{F}^n(\theta|v)}$. Also, $\Pr(G^n < (1-\delta)\bar{G}_n|Y_s^n(k_s+1) = \theta^n, v) \leq e^{-\frac{\delta^2}{2}\bar{G}_n}$ for any $\delta \in (0,1)$ by the Chernoff's inequality.[24] Similarly, $\bar{L}_n =$

[24] See Janson et al. (2011, Theorem 2.1).

$\frac{(n-1-k_s)\left(\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)-\bar{F}_s^n(\theta^n|v)\right)}{1-\bar{F}_s^n(\theta^n|v)}+1$ because there are $n-1-k_s$ bidders with signals below θ^n and the distribution of L^n is binomial and $\Pr(L^n<(1-\delta)\bar{L}_n|Y_s^n(k_s+1)=\theta,v)\le e^{-\frac{\delta^2}{2}\bar{L}_n}$. The random variable X^n and L^n are independent conditional on $Y_s^n(k_s+1)=\theta^n$. Moreover, $\Pr\left(b_p^n\, win|Y_s^n(k_s+1)=\theta^n,v\right)=\mathbb{E}[G^n/(L^n+G^n)|Y_s^n(k_s+1)=\theta^n,v]$. The function $G^n/(L^n+G^n)$ is concave in G^n and convex in L^n. Therefore, using Jensen's inequality and then the Chernoff bound we obtain

$$\mathbb{E}[\frac{G_n}{G_n+\bar{L}_n}|Y_s^n(k_s+1)=\theta^n,v]\le Q_n\le\mathbb{E}[\frac{\bar{G}_n}{\bar{G}_n+L_n}|Y_s^n(k_s+1)=\theta^n,v]$$
$$\frac{(1-\delta)\bar{G}_n}{\bar{G}_n(1-\delta)+\bar{L}_n}(1-e^{-\frac{\delta^2}{2}\bar{G}_n})\le Q_n\le\frac{\bar{G}_n}{\bar{G}_n+(1-\delta)\bar{L}_n}+e^{-\frac{\delta^2}{2}\bar{L}_n}.$$

where $Q_n=\Pr\left(b_p^n\, win|Y_s^n(k_s+1)=\theta^n,v\right)$. Our assumption $\lim F_s^n(\left[\underline{\theta}_p^n,\theta_p^n\right]|v)>0$ implies either $\bar{G}_n\to\infty$ or $\bar{L}_n\to\infty$ or both. Taking the limits and noting that δ is arbitrary we obtain $\lim\Pr\left(b_p^n\, win|Y_s^n(k_s+1)=\theta^n,v\right)=\lim\frac{\bar{G}_n}{\bar{G}_n+\bar{L}_n}$. Since $\bar{F}_s^n(\theta^n|v)\in(\kappa_s-\epsilon_1,\kappa_s+\epsilon_1)$ by assumption, we have

$$\lim\frac{\kappa_s\frac{\left(\kappa_s-\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s+\epsilon_1}}{\kappa_s\frac{\left(\kappa_s+\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s-\epsilon_1}+(1-\kappa_s)\frac{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)-\kappa_s+\epsilon_1}{1-\kappa_s-\epsilon_1}}\le\lim Q_n\le$$
$$\lim\frac{\kappa_s\frac{\left(\kappa_s+\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s-\epsilon_1}}{\kappa_s\frac{\left(\kappa_s-\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s+\epsilon_1}+(1-\kappa_s)\frac{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)-\kappa_s-\epsilon_1}{1-\kappa_s+\epsilon_1}}.$$

But $\lim\Pr\left(\bar{F}_s^n(Y_s^n(k_s+1)|v)\in(\kappa_s-\epsilon_1,\kappa_s+\epsilon_1)|v\right)=1$ for every $\epsilon_1>0$ by the LLN. Hence, $\lim\Pr\left(\bar{F}_s^n(Y_s^n(k_s+1)|v)\in(\kappa_s-\epsilon_1,\kappa_s+\epsilon_1)|Y_s^n(k_s+1)\in\left[\underline{\theta}_p^n,\theta_p^n\right],v\right)=1$. Therefore,

$$\lim\frac{\kappa_s\frac{\left(\kappa_s-\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s+\epsilon_1}}{\kappa_s\frac{\left(\kappa_s+\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s-\epsilon_1}+(1-\kappa_s)\frac{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)-\kappa_s+\epsilon_1}{1-\kappa_s-\epsilon_1}}\le\lim\Pr\left(b_p^n\, wins|Y_s^n(k_s+1)\in\left[\underline{\theta}_p^n,\theta_p^n\right],v\right)$$
$$\le\lim\frac{\kappa_s\frac{\left(\kappa_s+\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s-\epsilon_1}}{\kappa_s\frac{\left(\kappa_s-\epsilon_1-\bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s+\epsilon_1}+(1-\kappa_s)\frac{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)-\kappa_s-\epsilon_1}{1-\kappa_s+\epsilon_1}}.$$

Since this is true for each $\epsilon_1>0$, taking $\epsilon_1\to 0$ shows $\lim\Pr\left(b_p^n\, wins|P^n=b_p^n,v\right)=\lim\frac{\kappa_s-\bar{F}_s^n\left(\theta_p^n|v\right)}{F_s^n\left(\left[\underline{\theta}_p^n,\theta_p^n\right]|v\right)}$. □

Proof of items ii-v in Lemma B.1. Further below we argue that the expected number of losers at the pooling bid satisfies $0<\liminf\frac{\bar{L}^n}{\sqrt{n}}\le\limsup\frac{\bar{L}^n}{\sqrt{n}}<\infty$ if $\lim\sqrt{n}|F_s^n(\theta^n(v)|v)-F_s^n\left(\underline{\theta}_p^n|v\right)|<\infty$, and satisfies $0<\liminf\frac{\bar{L}^n}{nF_s^n\left(\left[\underline{\theta}_p^n,\theta^n(v)\right]|v\right)}\le\limsup\frac{\bar{L}^n}{nF_s^n\left(\left[\underline{\theta}_p^n,\theta^n(v)\right]|v\right)}\le 1$ if

$\lim \sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\underline{\theta}_p^n|v)| \to \infty$.

We will prove items *ii* and *iii* using these bounds for $\bar{L}^n$ items *iv* and *v* follow from an identical argument. We begin by proving the lower bounds in items *ii* and *iii*. Note that $\Pr(L^n \geq \bar{L}^n - 1 | P^n = b_p^n, v) \geq 1/2$.[25]

$$\begin{aligned}
\Pr(b_p^n\, lose | P^n = b_p^n, v) &\geq \mathbb{E}\left[\frac{L^n}{X} | L^n \geq \bar{L}^n - 1, P^n = b_p^n, v\right] \Pr(L^n \geq \bar{L}^n - 1 | P^n = b_p^n, v) \\
&\geq \mathbb{E}\left[\frac{\bar{L}^n - 1}{X} | L^n \geq \bar{L}^n - 1, P^n = b_p^n, v\right] \frac{1}{2} \\
&\geq \frac{(\bar{L}^n - 1)/2}{\mathbb{E}[X^n | L^n \geq \bar{L}^n - 1, P^n = b_p^n, v]} \text{ (by Jensen's Inequality)}
\end{aligned}$$

Note that $\mathbb{E}[X^n | L^n \geq \bar{L}^n - 1, P^n = b_p^n, v] \Pr(L^n \geq \bar{L}^n - 1, P^n = b_p^n | v) \leq \mathbf{E}[X^n | v] = nF_s^n([\underline{\theta}_p^n, \theta_p^n] | v)$. Therefore,

$$\begin{aligned}
\Pr(b_p^n\, lose | P^n = b_p^n, v) &\geq \frac{(\bar{L}^n - 1)}{nF_s^n([\underline{\theta}_p^n, \theta_p^n] | v)} \frac{\Pr(L^n \geq \bar{L}^n - 1 | P^n = b_p^n, v) \Pr(P^n = b_p^n | v)}{2} \\
\Pr(b_p^n\, lose | P^n = b_p^n, v) nF_s^n([\underline{\theta}_p^n, \theta_p^n] | v) &\geq (\bar{L}^n - 1) \frac{\Pr(P^n = b_p^n | v)}{4}
\end{aligned}$$

Taking limits and substituting $0 < \liminf \frac{\bar{L}^n - 1}{\sqrt{n}} < \limsup \frac{\bar{L}^n - 1}{\sqrt{n}} < \infty$ if $\lim \sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\underline{\theta}_p^n|v)| < \infty$;and

$$0 < \liminf \frac{\bar{L}^n - 1}{nF_s^n([\underline{\theta}_p^n, \theta^n(v)] | v)} \leq \limsup \frac{\bar{L}^n - 1}{nF_s^n([\underline{\theta}_p^n, \theta^n(v)] | v)} \leq 1$$

if $\lim \sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\underline{\theta}_p^n|v)| \to \infty$ delivers the lower bounds in items *ii* and *iii*.

We now establish the upper bounds in items *ii* and *iii*. If $\lim \sqrt{n} F_s^n([\underline{\theta}_p^n, \theta_p^n] | v) \in (0, \infty)$, then $\lim \sqrt{n}|F_s^n(\theta^n(v)|V = v) - F_s^n(\underline{\theta}_p^n|V = v)| < \infty$ (because $\lim \Pr(P^n = b_p^n | V = v) > 0$) and the upper bound in item *ii* is trivially satisfied. Suppose $\lim \sqrt{n} F_s^n([\underline{\theta}_p^n, \theta_p^n] | v) = \infty$. Pick $\delta \in (0,1)$ and let $\bar{Y}^n = nF_s^n([\underline{\theta}_p^n, \theta_p^n] | v)$. Then

$$\begin{aligned}
\Pr[b_p^n\, lose | P^n = b_p^n, v] \leq{}& \frac{\mathbb{E}[L^n | X^n > (1-\delta)\bar{Y}^n, P^n = b_p^n, v] \Pr(X^n > (1-\delta)\bar{Y}^n | P^n = b_p^n, v)}{(1-\delta)\bar{Y}^n} \\
&+ \Pr(X^n \leq \bar{Y}^n (1-\delta) | P^n = b_p^n, v).
\end{aligned}$$

[25] Conditional on $Y_s^n(k_s + 1) = \theta \in [\underline{\theta}_p^n, \theta_p^n]$ and $V = v$, the number of losers L^n is a binomial random variable. The median of the binomial differs from the mean by at most one. Therefore, $\Pr(L^n \geq \mathbf{E}[L^n | Y_s^n(k_s + 1) = \theta, v] - 1 | Y_s^n(k_s + 1) = \theta, v) \geq 1/2$. In turn, this implies that $\Pr(L^n \geq \bar{L}^n - 1 | P^n = b_p^n, v) \geq 1/2$.

However, $\mathbb{E}[L^n|X^n>(1-\delta)\bar{Y}^n,P^n=b_p^n,v]\Pr\left(X^n>(1-\delta)\bar{Y}^n|P^n=b_p^n,v\right)\leq\bar{L}^n$. Therefore,

$$\frac{\Pr\left[b_p^n\, lose|P^n=b_p^n,v\right]\bar{Y}^n}{\bar{L}^n}\leq\frac{1}{1-\delta}+\frac{\bar{Y}^n}{\bar{L}^n}\Pr\left(X^n\leq\bar{Y}^n(1-\delta)|P^n=b_p^n,v\right)$$

Chernoff's inequality implies that $\lim\Pr(X^n\leq(1-\delta)\bar{Y}^n|v)<\exp(-\frac{\delta^2\bar{Y}^n}{3})$ and hence $\Pr(X^n\leq\bar{Y}^n(1-\delta)|P^n=b_p^n,v)\leq\frac{\exp\left(-\frac{\delta^2Y^n}{3}\right)}{\Pr\left(P^n=b_p^n|v\right)}$. Therefore, $\lim\frac{\Pr\left[b_p^n\, lose|P^n=b_p^n,v\right]\bar{Y}^n}{\bar{L}^n}\leq\frac{1}{1-\delta}$. Substituting for the number of losers $\bar{L}^n$ now delivers the upper bounds in items i and ii.

We now show that $0<\liminf\frac{\bar{L}^n}{\sqrt{n}}\leq\limsup\frac{\bar{L}^n}{\sqrt{n}}<\infty$ if $\lim\sqrt{n}|F_s^n(\theta^n(v)|v)-F_s^n(\underline{\theta}_p^n|v)|<\infty$, and $0<\liminf\frac{\bar{L}^n}{nF_s^n([\underline{\theta}_p^n,\theta^n(v)]|v)}\leq\limsup\frac{\bar{L}^n}{nF_s^n([\underline{\theta}_p^n,\theta^n(v)]|v)}\leq1$ if $\lim\sqrt{n}|F_s^n(\theta^n(v)|v)-F_s^n\left(\underline{\theta}_p^n|v\right)|\to\infty$.

Pick any $\theta^n\in\left[\underline{\theta}_p^n,\theta_p^n\right]$ and let $a(\theta^n):=\bar{F}_s^n(\theta^n(v)|v)-\bar{F}_s^n(\theta^n|v)=\kappa_s-\bar{F}_s^n(\theta^n|v)$. Recall that $\Pr(L^n=i|Y_s^n(k_s+1)=\theta^n,v)=bi(i;n-1-k_s,p^n)$ and $\mathbb{E}[L^n|Y_s^n(k_s+1)=\theta^n,v]=np(a(\theta^n))\left(1-\kappa_s-\frac{1}{n}\right)$ where $p(a(\theta^n))=\frac{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)-\kappa_s+a(\theta^n)}{1-\kappa_s+a(\theta^n)}$. Calculating the number of losers we find $\bar{L}^n=-\left(1-\kappa_s-\frac{1}{n}\right)\int_{\underline{a}^n}^{\bar{a}^n}np(a)d\Lambda(a)$ where $\underline{a}^n=a(\underline{\theta}_p^n)$, $\bar{a}^n=a(\theta_p^n)$, and $\Lambda(a):=\Pr\left(F_s^n(Y_s^n(k_s+1)|v)-\kappa_s\geq a|P=b_p^n,v\right)$. Integrating by parts and substituting $p(\underline{a}^n)=0$, $\Lambda(\bar{a}^n)=0$, and $p'(a)=\frac{\left(1-\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)\right)}{(1-\kappa_s+a)^2}=\frac{1-\kappa_s+\underline{a}^n}{(1-\kappa_s+a)^2}$ delivers $\bar{L}^n/n=\left(1-\kappa_s-\frac{1}{n}\right)(1-\kappa_s+\underline{a}^n)\int_{\underline{a}^n}^{\bar{a}^n}\frac{\Lambda(a)}{(1-\kappa_s+a)^2}da$. Hence,

$$C^n\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da\leq\bar{L}^n/n\leq\frac{1-\kappa_s}{1-\kappa_s+\underline{a}^n}\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da\leq\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da.$$

where $C^n=\frac{(1-\kappa_s)(1-\kappa_s+\underline{a}^n)}{(1-\kappa_s+\bar{a}^n)^2}$.

Pick any $\epsilon>0$ and let a_*^n be such that $\Pr(F_s^n(Y_s^n(k_s+1)|v)-\kappa_s\geq a^n|P^n=b_p^n,v)=1-\epsilon$. The central limit theorem implies that $\lim\sqrt{n}a_*^n\in(0,\infty)$. Moreover, $\lim\sqrt{n}(a^n-\underline{a}^n)>0$ because $\Pr(F_s^n(Y_s^n(k_s+1)|v)-\kappa_s\leq a^n|P^n=b_p^n,v)=\epsilon$ for each n. Therefore,

$$\begin{aligned}
\int_{\underline{a}^n}^{a_*^n}\Lambda(a)da&\leq\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da\leq\max\{-\underline{a}^n,0\}+\int_0^{\bar{a}^n}\Lambda(a)da\\
\int_{\underline{a}}^{a_*^n}(1-\epsilon)da&\leq\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da\leq\max\{-\underline{a}^n,0\}+\frac{\int_0^{\bar{a}^n}e^{-\frac{a^2n}{2}}d\theta}{\Pr\left(P^n=b_p^n|v\right)}\text{ (by Chernoff's inequality)}\\
(1-\epsilon)(a_*^n-\underline{a}^n)&\leq\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da=\max\{-\underline{a}^n,0\}+\frac{\sqrt{2}erf(\sqrt{n}\bar{a})}{\Pr\left(P^n=b_p^n|v\right)\sqrt{\pi n}}\\
&\leq\max\{-\underline{a}^n,0\}+\frac{1}{\Pr\left(P^n=b_p^n|v\right)\sqrt{2\pi n}}
\end{aligned}$$

where $erf(x)=\frac{1}{\sqrt{\pi}}\int_0^xe^{-t^2}dt\in[0,1/2]$ is the error function.

Note $-\underline{a}^n=F_s^n(\theta^n(v)|V=v)-F_s^n\left(\underline{\theta}_p^n|V=v\right)$. Suppose that $-\lim\sqrt{n}\underline{a}^n<\infty$.

If $-\lim\sqrt{n}\underline{a}^n = \delta_1 < \infty$, then $\lim\sqrt{n}(a_*^n - \underline{a}^n) = \delta \in (0,\infty)$. The fact that $\frac{\bar{L}^n}{\sqrt{n}} \in \left(\sqrt{n}C^n\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da, \sqrt{n}\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da\right)$ and the bounds for $\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da$ together imply that

$$C^n\left((1-\epsilon)\sqrt{n}(a_*^n-\underline{a}^n)\right) \leq \frac{\bar{L}^n}{\sqrt{n}} \leq \max\{\sqrt{n}\underline{a}^n, 0\} + \frac{1}{\Pr\left(P^n = b_p^n|v\right)\sqrt{2\pi}}$$

$$0 < (1-\epsilon)C\delta \leq \liminf\frac{\bar{L}^n}{\sqrt{n}} \leq \limsup\frac{\bar{L}^n}{\sqrt{n}} \leq \max\{\delta_1, 0\} + \frac{1}{\Pr\left(P^n = b_p^n|v\right)\sqrt{2\pi}} < \infty$$

where $C = \liminf C^n$.

If $-\lim\sqrt{n}\underline{a}^n = \infty$, then $L^n \in \left(n\sqrt{n}C^n\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da, n\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da\right)$ and the bounds for $\int_{\underline{a}^n}^{\bar{a}^n}\Lambda(a)da$ together imply that

$$C^n\left(\frac{(1-\epsilon)n(a_*^n-\underline{a}^n)}{-n\underline{a}^n}\right) \leq \frac{\bar{L}^n}{-n\underline{a}^n} \leq 1 - \frac{1}{\Pr\left(P^n = b_p^n|v\right)\underline{a}^n\sqrt{2\pi n}}$$

$$\lim C^n\left((1-\epsilon)\left(\frac{\sqrt{n}a_*^n}{-n\underline{a}^n} + 1\right)\right) \leq \liminf\frac{\bar{L}^n}{-n\underline{a}^n} \leq \limsup\frac{\bar{L}^n}{-n\underline{a}^n} \leq 1$$

$$0 < C(1-\epsilon) \leq \liminf\frac{\bar{L}^n}{-n\underline{a}^n} \leq \limsup\frac{\bar{L}^n}{-n\underline{a}^n} \leq 1.$$

□

Proof of the calculation for the case where $\lim\Pr\left(P^n \geq b_p^n|v\right) = 0$ *in Lemma B.1.* As before, let X^n denote the random variable which is equal to the number of bidders in the interval $\left[\underline{\theta}_p^n, \theta_p^n\right]$. Redefine L^n to denote the random variable which is equal to the number of losers with signals that exceed $\underline{\theta}_p^n$. Note that $\mathbb{E}\left[L^n|Y_s^n(k+1) \geq \underline{\theta}_p^n, V=v\right] = \mathbb{E}\left[L^n|L^n \geq 1, V=v\right]$. Pick a $\delta > 0$, and let $d^n = (1-\delta)k_s\frac{F_s^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right)}{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right)}$ and observe that $\lim\frac{d^n}{\sqrt{n}} > 0$. We will show $\lim\frac{\mathbb{E}\left[\frac{L^n}{X^n}|L^n \in [1,d^n], V=0\right]}{\frac{\bar{F}_s^n(\underline{\theta}_p^n|V=v)\left(1-\bar{F}_s\left(\underline{\theta}_p|V=v\right)\right)}{nF_s^n([\underline{\theta}_p^n,\theta_p^n]|V=v)\left(\kappa_s-\bar{F}_s\left(\underline{\theta}_p|V=v\right)\right)}} = 1$ and $\lim\frac{\Pr\left(b_p^n\, loses|P^n=b_p^n, V=v\right)}{\mathbb{E}\left[\frac{L^n}{X^n}|L^n\in[1,d^n], V=0\right]} = 1$.

Step 1. $\lim\frac{\mathbb{E}[L^n|L^n\in[1,d^n],v]}{a^n} = 1$, where $a^n = \frac{\kappa_s\left(1-\bar{F}_s^n\left(\underline{\theta}_p^n|V=v\right)\right)}{\kappa_s-\bar{F}_s^n\left(\underline{\theta}_p^n|V=v\right)}$. Note $\lim\mathbb{E}[L^n|L^n\in[1,d^n],v] = \frac{\sum_{i=1}^{d^n} i\,bi(k_s+i,n;p^n)}{\sum_{i=1}^{d^n} bi(k_s+i,n;p^n)}$ where $p^n = \bar{F}_s^n(\underline{\theta}_p^n|v)$. Observe that

$$\frac{bi(k+i,n;p^n)}{bi(k+i,n;\kappa_s)}bi(k+i,n;\kappa_s) = bi(k+i,n;\kappa_s)\left(\frac{p^n}{\kappa_s}\right)^{k_s}\left(\frac{1-p^n}{1-\kappa_s}\right)^{n-k_s}\left(\frac{p^n(1-\kappa_s)}{\kappa_s(1-p^n)}\right)^i.$$

Therefore $\mathbb{E}[L^n|L^n\in[1,d^n],v] = \frac{\sum_{i=1}^{d^n} i\,r(n)^i bi(k_s+i,n;\kappa_s)}{\sum_{i=1}^{d^n} r(n)^i bi(k_s+i,n;\kappa_s)}$ where $r(n) = \frac{p^n(1-\kappa_s)}{\kappa_s(1-p^n)} < 1$. Pick any

$J<d^n$. For each $i<J$,

$$(1-\epsilon^n)\phi\left(\frac{J}{\sqrt{n(1-\kappa_s)\kappa_s}}\right)\leq\sqrt{n(1-\kappa_s)\kappa_s}bi(k+i,n;\kappa_s)\leq(1+\epsilon^n)\phi(0)$$

by the local limit theorem (Proposition A.2). Hence,

$$(1-\epsilon^n)\frac{\phi\left(\frac{J}{\sqrt{n}}\right)\sum_{i=1}^{J}ir(n)^i}{\phi(0)\sum_{i=1}^{d^n}r(n)^i}\leq\frac{\sum_{i=1}^{d^n}ir^i\sqrt{n}bi(k+i,n;\kappa_s)}{\sum_{i=1}^{d^n}r^i\sqrt{n}bi(k+i,n;\kappa_s)}\leq\frac{\phi(0)}{\phi\left(\frac{J}{\sqrt{n}}\right)}\frac{\sum_{i=1}^{d^n}ir(n)^i}{\sum_{i=1}^{J}r(n)^i}(1+\epsilon^n).$$

Evaluating the geometric series we find

$$\frac{\phi\left(\frac{J}{\sqrt{n}}\right)(1-\epsilon^n)}{\phi(0)\left(1-r(n)^{d^n}\right)}\left(\frac{1-r(n)^J}{1-r(n)}-Jr(n)^J\right)\leq Q\leq\frac{\phi(0)(1+\epsilon^n)}{\phi\left(\frac{J}{\sqrt{n}}\right)\left(1-r(n)^J\right)}\left(\frac{1-r(n)^{d^n}}{1-r(n)}-d^nr(n)^{d^n}\right)$$
$$\frac{\phi\left(\frac{J}{\sqrt{n}}\right)(1-\epsilon^n)}{\phi(0)\left(1-r(n)^{d^n}\right)}\left(\frac{1-r(n)^J}{1-r(n)}-Jr(n)^J\right)\leq Q\leq\frac{\phi(0)(1+\epsilon^n)}{\phi\left(\frac{J}{\sqrt{n}}\right)\left(1-r(n)^J\right)}$$

where $Q=\mathbb{E}[L^n|L^n\in[1,d^n],v]$.

Case 1. $\bar{F}(\underline{\theta}_p|v)<\kappa_s$. In this case, $\lim r(n)=r<1$. Picking $J=n^{1/4}<d^n$ and taking the limit as $n\to\infty$ we find $\lim\mathbb{E}[L^n|L^n\in[1,d^n],v_i]=\frac{1}{1-r}=\frac{\kappa_s\left(1-\bar{F}_s\left(\underline{\theta}_p|V=v\right)\right)}{\kappa_s-\bar{F}_s\left(\underline{\theta}_p|V=v\right)}=\lim a^n$.

Case 2. $\overline{F}_s\left(\underline{\theta}_p|v_i\right)=\kappa_s$. In this case $r(n)<1$ for all n sufficiently large but $\lim r(n)=1$. Note that $\lim\frac{1-r(n)}{1/a^n}=1$. For any constant m, $ma^n<d^n$ for sufficiently large n because $d^n/a^n\to\infty$. Substituting $1/a^n$ for $1-r(n)$ and setting $J=ma^n$ for any arbitrary m we find

$$\frac{\phi(ma^n/\sqrt{n})}{\phi(0)}\frac{a^n\left(1-(1-1/a^n)^{ma^n}\right)-ma^n(1-1/a^n)^{ma^n}}{1-(1-1/a^n)^{d^n}}\frac{1-\epsilon^n}{a^n}\leq X\leq\frac{\phi(0)}{\phi(ma^n/\sqrt{n})}a^n\frac{1+\epsilon^n}{a^n}$$
$$\frac{\phi(ma^n/\sqrt{n})}{\phi(0)}\frac{\left(1-(1-1/a^n)^{ma^n}\right)-m(1-1/a^n)^{ma^n}}{1-(1-1/a^n)^{d^n}}(1-\epsilon^n)\leq X\leq\frac{\phi(0)}{\phi(ma^n/\sqrt{n})}(1+\epsilon^n)$$

where $X=\frac{\mathbb{E}[L^n|L^n\in[1,d^n],v_i]}{a^n}$. Taking the limit as $n\to\infty$ and noting that $a^n\to\infty$, $a^n/\sqrt{n}\to0$ and $d^n/a^n\to\infty$ we obtain $(1-1/a^n)^{ma^n}\to\exp(-m)$, $\phi(ma^n/\sqrt{n})\to\phi(0)$, and $(1-1/a^n)^{d^n}\to0$. Therefore $1-\exp(-m)-\exp(-m)m\leq\lim\mathbb{E}[L^n|L^n\in[1,d^n],v_i]/a^n\leq1$. As m is arbitrary, taking the limit as $m\to\infty$ we find $\mathbb{E}[L^n|L^n\in[1,d^n],v_1]/a^n\to1$.

Step 2. We show $\Pr(L^n>d^n|L^n\geq1,v)\leq A\exp(-d^n/a^n)\to0$ and $\Pr[Y^n(k+1)>\theta_p^n|Y^n(k+1)]>\underline{\theta}_p^n,v\leq A\exp(-d^n/a^n)\to0$ where A is an arbitrary positive constant.

Following the procedure from the previous step, we find

$$\Pr(L^n > d^n | L^n \geq 1, v) = \frac{\sum_{i=d^n}^{n-k} bi(k+i,n;\kappa)}{\sum_{i=1}^{n-k} bi(k+i,n;\kappa)}$$
$$= \frac{r^{d^n}(1-\epsilon_1^n)\left(a^n + \frac{(1-1/a^n)^n}{a^n} - (1-1/a^n)^n\right)}{(1-\epsilon_2^n)\left(a^n + \frac{(1-1/a^n)^n}{a^n} - (1-1/a^n)^n\right)} \leq A\exp(-d^n/a^n)$$

where last inequality is a consequence of the fact that $(1-1/a^n)^{d^n}$ is of the order of $\exp(-d^n/a^n)$. Also, we have

$$\Pr\left[Y^n(k+1) > \theta_p^n | Y^n(k+1) > \underline{\theta}_p^n, v\right] =$$
$$\Pr(L^n \in [1,d^n] | L^n > 1, v) \Pr(X^n < L^n | L^n \in [1,d^n], L^n > 1, v)$$
$$+ \Pr(L^n > d^n | L^n > 1, v) \Pr(X^n < L^n | L^n > d^n, L^n > 1, v).$$

Consequently,

$$\Pr\left[Y^n(k+1) > \theta_p^n | Y^n(k+1) > \theta_p^n, v\right] \leq \Pr(L^n > d^n | L^n > 1, v) + \Pr(X^n < L^n | L^n \in [1,d^n], L^n > 1, v)$$
$$\leq \frac{\sum_{i=d^n}^{n-k} bi(k+i,n;\kappa)}{\sum_{i=1}^{n-k} bi(k+i,n;\kappa)} + \exp\left(--\delta^2 d^n/2\right)$$
$$\leq A\exp(-d^n/a^n) + \exp\left(-\delta^2 d^n/2\right)$$
$$\leq A\exp(-d^n/a^n)$$

where in the last inequality we use the fact that $A\exp(-d^n/a^n) \geq \exp(-\delta^2 d^n/2)$ and redefine the constant A without changing the order of the term.

Step 3. We now show

$$\frac{\overline{F}_s^n\left(\underline{\theta}_p^n | v\right)}{(1+\delta)F^n\left([\underline{\theta}_p^n, \theta_p^n] | v\right)} \frac{\mathbb{E}[L^n | L^n \in [1,d^n], v]}{k_s} \leq \Pr(b^p\, loses | L^n \geq 1, v) \leq$$
$$\frac{\overline{F}_s^n\left(\underline{\theta}_p^n | v\right)}{(1-\delta)F^n\left([\underline{\theta}_p^n, \theta_p^n] | v\right)} \frac{\mathbb{E}[L^n | L^n \in [1,d^n], v]}{k_s} + A\exp(-d^n/a^n)$$

We first give a lower bound for the probability of losing:

$$\Pr(b^p\, loses | L^n \geq 1, v) \geq \mathbb{E}\left[\min\left[\frac{L^n}{X^n}, 1\right] | L^n \in [1,d^n], v\right] \Pr(L^n \in [1,d^n] | L^n \geq 1, v)$$

Note that $\Pr(L^n \in [1,d^n] | L^n \geq 1,v) \to 1$, thus

$$\Pr(b^p\, loses | L^n \geq 1,v) \geq \mathbb{E}\left[\min\left[\frac{L^n}{X^n},1\right] | L^n \in [1,d^n],v\right](1-\delta_1)$$

where δ_1 is an arbitrarily small constant. The facts that $\min[L^n/X^n,1]$ is a concave function of X^n and Jensen's inequality together imply that

$$\mathbb{E}[\min[L^n/X^n,1] | L^n \in [1,d^n],v] \geq \mathbb{E}\left[\min\left[\frac{L^n}{\mathbb{E}[X^n|L^n,v]},1\right] | L^n \in [1,d^n],v\right].$$

By definition $\mathbb{E}[X^n|L^n,v_i] > d^n$, therefore

$$\begin{aligned}\mathbb{E}\left[\min\left[\frac{L^n}{\mathbb{E}[X^n|L^n,v_i]},1\right] | L^n \in [1,d^n],v\right] &= \mathbb{E}\left[\frac{L^n}{\mathbb{E}[X^n|L^n,v_i]} | L^n \in [1,d^n],v\right] \\ &= \frac{\overline{F_s}^n\left(\underline{\theta}_p^n|v\right)}{F_s^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right)} \mathbb{E}\left[\frac{L^n}{L^n+k_s} | L^n \in [1,d^n],v\right]\end{aligned}$$

Noticing that $\frac{L^n}{L^n+k}$ is a concave function of L and applying Jensen's inequality implies that

$$\begin{aligned}\mathbb{E}[\min[L^n/X^n,1] | L^n \in [1,d^n],v] &\geq \frac{\overline{F_s}^n\left(\underline{\theta}_p^n|v\right)}{F_s^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right)} \frac{\frac{\mathbb{E}[L^n|L^n \in [1,d^n],v]}{k_s}}{\frac{\mathbb{E}[L^n|L^n \in [1,d^n],v]}{k_s}+1} \\ &\geq \frac{\overline{F_s}^n\left(\underline{\theta}_p^n|v\right)}{F_s^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right)} \frac{\frac{\mathbb{E}[L^n|L^n \in [1,d^n],v]}{k_s}}{1+\delta_2}\end{aligned}$$

where $\delta_2 := \mathbb{E}[L^n|L^n \in [1,d^n],v]/k$ is an arbitrary positive constant. Note that $\mathbb{E}[L^n|L^n \in [1,d^n],v]/k \to 0$, therefore we can choose δ_2 arbitrarily small for large n. Therefore,

$$\Pr(b^p\, loses | L^n \geq 1,v) \geq (1-\delta) \frac{\overline{F_s}^n\left(\underline{\theta}_p^n|v\right)}{F_s^n\left([\underline{\theta}_p^n,\theta_p^n]|v\right) k_s} \mathbb{E}[L^n|L^n \in [1,d^n],v]$$

where $1-\delta = \min\{1/(1+\delta_2), 1-\delta_1\}$.

We now provide an upper bound for the probability of losing:

$$
\begin{aligned}
&\Pr(b^p\, loses|L^n \geq 1,v) \leq \\
&\qquad \mathbb{E}[\min[L^n/X^n,1]|L^n \in [1,d^n],v] + \Pr(L^n > d^n|L^n \geq 1,v) \leq \\
&\frac{\overline{F}^n_s\left(\underline{\theta}^n_p|v\right)}{(1-\delta)F^n_s\left([\underline{\theta}^n_p,\theta^n_p]|v\right)}\mathbb{E}\left[\frac{L^n}{L^n+k_s}|L^n \in [1,d^n],v\right] + \Pr(L^n > d^n|L^n \geq 1,v) + \exp\left(-\delta^2 d^n/3\right) \leq \\
&\qquad \frac{\overline{F}^n_s\left(\underline{\theta}^n_p|v\right)}{(1-\delta)F^n_s\left([\underline{\theta}^n_p,\theta^n_p]|v\right)}\frac{\mathbb{E}[L^n|L^n \in [1,d^n],v]}{k_s} + A\exp(-d^n/a^n) + \exp\left(-\delta^2 d^n/3\right) \leq \\
&\qquad\qquad \frac{\overline{F}^n_s\left(\underline{\theta}^n_p|v\right)}{(1-\delta)F^n\left([\underline{\theta}^n_p,\theta^n_p]|v\right)}\frac{\mathbb{E}[L^n|L^n \in [1,d^n],v]}{k_s} + A\exp(-d^n/a^n).
\end{aligned}
$$

the first inequality follows because $\mathbb{E}[X^n|L^n = i \in [1,d^n],v] = (k_s+i)\frac{F^n_s\left([\underline{\theta}^n_p,\theta^n_p]|v\right)}{\overline{F}^n_s\left(\underline{\theta}^n_p|v\right)}$ is less than $(1-\delta)(k_s+i)\frac{F^n_s\left([\underline{\theta}^n_p,\theta^n_p]|v\right)}{\overline{F}^n_s\left(\underline{\theta}^n_p|v\right)}$ with probability $\exp(-\delta^2 d^n/3)$ by Chernoff's inequality and the second follows because we showed that $\Pr(L^n > d^n|L^n \geq 1,v) \leq A\exp(-d^n/a^n)$ in step 2. To obtain the last inequality we use the fact $A\exp(-d^n/a^n) > \exp(-\delta^2 d^n/3)$ and redefine the constant A without changing the order of the term. The lemma now follows as $\frac{d^n}{a^n}\exp(-d^n/a^n) \to 0$ because $d^n/a^n \to \infty$ and because the constants δ are arbitrary. $\square$

Lemma B.2. *Fix a sequence of bidding equilibria* $\mathbf{H}$ *and suppose that* $\lim\sqrt{n}|\bar{F}^n_s(\theta^n_s(1)|V = v) - \bar{F}^n_s(\theta^n_s(0)|V = v)| \to \infty$. *If there is pooling by pivotal types, then* $\lim\Pr(P^n \leq b^n_p|V = 1) = 1$ *and* $\lim\Pr(P^n < b^n_p|V = 0) = 0$.

Proof. Pooling by pivotal types implies that $\lim\Pr\left(P^n = b^n_p|V = v\right) > 0$ for $v = 0,1$. Suppose $\lim\Pr(P^n < b^n_p|V = 0) > 0$ then $\lim\sqrt{n}\left(F^n_s(\theta_s(0)|0) - F^n_s\left(\underline{\theta}^n_p|0\right)\right) \in (-\infty,\infty)$. Moreover, $\lim\Pr\left(P^n = b^n_p|V = 1\right) > 0$ and $\lim\sqrt{n}|\bar{F}^n_s(\theta^n_s(1)|V = 1) - \bar{F}^n_s(\theta^n_s(0)|V = 1| \to \infty$ together imply $\lim\sqrt{n}\left(F^n_s(\theta_s(1)|1) - F^n_s\left(\underline{\theta}^n_p|1\right)\right) = \infty$. Along any sequence where the limit in the equation below exists, Lemma A.2 implies that there is a constant C such that

$$
\lim\frac{\Pr\left(b^n\, lose|P^n = b^n_p,V = 0\right)}{\Pr\left(b^n\, lose|P^n = b^n_p,V = 1\right)} = \leq \frac{1}{\eta}\lim\frac{C}{\sqrt{n}\left(F^n_s(\theta^n_s(1)|1) - F^n_s\left(\underline{\theta}^n_p|1\right)\right)} = 0
$$

showing that pooling is not possible. Therefore, if there is pooling by pivotal types, then $\lim\Pr(P^n < b^n_p|V = 0) = 0$.

Suppose $\lim\Pr(P^n \leq b^n_p|V = 1) < 1$. Then $\lim\sqrt{n}\left(F^n_s\left(\theta^n_p|1\right) - F^n_s(\theta^n_s(1)|1)\right) \in (-\infty,\infty)$. Moreover, $\lim\Pr\left(P^n = b^n_p|V = 0\right) > 0$ and $\lim\sqrt{n}|\bar{F}^n_s(\theta^n_s(1)|V = 1) - \bar{F}^n_s(\theta^n_s(0)|V = 1| \to \infty$ together imply $\lim\sqrt{n}\left(F^n_s\left(\theta^n_p|0\right) - F^n_s(\theta^n_s(0)|0)\right) = \infty$. Using

Lemma A.2 we obtain

$$\lim \frac{\Pr\left(b^n\, win|P^n = b_p^n, V=1\right)}{\Pr\left(b^n\, win|P^n = b_p^n, V=0\right)} \leq C \lim \frac{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|0\right)}{F_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|1\right)} \frac{\frac{1}{\sqrt{n}}}{\left(F_s^n\left(\theta_p^n|0\right) - F_s^n\left(\theta_s^n(0)|0\right)\right)} = 0$$

again showing that pooling is not possible. Therefore, if there is pooling by pivotal types, then $\lim \Pr(P^n \leq b_p^n|V=1) = 1$. $\square$

B.2. Existence Proofs of Section 4

Proof of Step 2 of Proposition 4.1. Pick $\theta' \in [1/3, 2/3]$, $\theta'' \in [2/3, 1]$ and let $\tilde{\theta} = (\theta', \theta'')$. Suppose that $\theta \in [0, \theta') \cup (2/3, \theta'']$ select market s and all others select market r. The expected payoff of a type $\theta \in \mathcal{E}(1)$ who submits a bid equal to $b=1$ in market s or in market r is given by $u_{\tilde{\theta}}(s|\mathcal{E}(1)) = G_s(1/3|1) + \int_{1/3}^{\theta'}(1 - b_s^n(\theta))dG_s(\theta|1)$ and $u_{\tilde{\theta}}(r|\mathcal{E}(1)) = G_r(\theta'|1)(1-c) + \int_{\theta'}^{2/3}(1 - b_r^n(\theta))dG_r(\theta|1)$ where $G_m(\theta|v) = \Pr(Y_m^{n-1}(k_m) \leq \theta|V=v)$.[26] The expected payoff of type 1/3 (hence the payoff for any $\theta \in \mathcal{E}(1/2)$) that submits a bid equal to $b = b_s^n(1/3)$ in market s and the expected payoff of type θ' that submits a bid equal to $b = b_r^n(\theta')$ in market r are given by $u_{\tilde{\theta}}(s|\mathcal{E}(1/2)) = G_s(1/3|1)/2$ and $u_{\tilde{\theta}}(r|\mathcal{E}(1/2)) = G_r(\theta'|1)(1-c)/2 - G_r(\theta'|0)c/2$. Notice that $\Pr(Y_s^{n-1}(k_s) \leq \theta|V=1)$ and $\Pr(Y_s^{n-1}(k_s) \leq \theta|V=0)$ are binomial distributions with parameters $\bar{F}_s([\theta, 2/3]|1) + \bar{F}_s([2/3, \theta'']|1)$ and $\bar{F}_s([\theta, 2/3]|0)$. Therefore, the functions $\mathbb{E}[V|Y_m^{n-1}(k_m) = \theta]$, $G_m(\theta|v)$, and $dG_m(\theta|v)$ are continuous in θ' and θ''.

Let $\tilde{\theta} = (\theta', \theta'') \in [1/3, 2/3] \times [2/3, 1]$ and define

$$\Gamma_1(\tilde{\theta}) = \begin{cases} [\frac{1}{3}, \frac{2}{3}] & \text{if } u_{\tilde{\theta}}(s|\mathcal{E}(1/2)) = u_{\tilde{\theta}}(r|\mathcal{E}(1/2)) \\ \frac{2}{3} & \text{if } u_{\tilde{\theta}}(s|\mathcal{E}(1/2)) > u_{\tilde{\theta}}(r|\mathcal{E}(1/2)) \\ \frac{1}{3} & \text{if } u_{\tilde{\theta}}(s|\mathcal{E}(1/2)) < u_{\tilde{\theta}}(r|\mathcal{E}(1/2)) \end{cases}$$

and

$$\Gamma_2(\tilde{\theta}) = \begin{cases} [\frac{2}{3}, 1] & \text{if } u_{\tilde{\theta}}(r|\mathcal{E}(1)) = u_{\tilde{\theta}}(r|\mathcal{E}(1)) \\ 1 & \text{if } u_{\tilde{\theta}}(s|\mathcal{E}(1)) > u_{\tilde{\theta}}(r|\mathcal{E}(1)) \\ 2/3 & \text{if } u_{\tilde{\theta}}(s|\mathcal{E}(1)) < u_{\tilde{\theta}}(r|\mathcal{E}(1)). \end{cases}$$

The correspondence $\Gamma = \Gamma_1 \times \Gamma_2$ is UHC, convex valued, compact valued and therefore has a fixed point (θ_1, θ_2) and this fixed point is an equilibrium. The fixed point is an equilibrium because the correspondence Γ is defined so that all types $\theta \in \mathcal{E}(1/2)$

[26] If no types $\theta \in \mathcal{E}(1) \cup \mathcal{E}(1/2)$ bid in market s, then $\mathbb{E}\left[V|Y_s^{n-1}(k_s) = \theta\right]$ is not well defined. In this case any bid $b > 0$ is optimal for $\theta \in \mathcal{E}(1/2)$ (and similarly in market r). Although this situation never occurs in equilibrium, for completeness we assume that $\mathbb{E}\left[V|Y_m^{n-1}(k_m) = \theta\right] = 1/2$ in this case.

choose the market that gives them the highest payoff and if $\theta_1 \in (1/3, 2/3)$, then type θ_1 as well as all types $\theta \in \mathcal{E}(1/2)$ are indifferent between the two markets. The situation is similar for types $\theta \in \mathcal{E}(1)$ and all $\theta \in \mathcal{E}(0)$ choose market s by construction. Moreover, conditional on these choices, the bidding function $\mathbb{E}[V|Y_m^{n-1}(k_m)=\theta]$ is a bidding equilibrium in market m, and this bidding function delivers the payoffs used to construct the correspondence Γ. $\square$

Proof of Example 4.1. Pick $\epsilon < \frac{1-2c}{2}$ and recall that $b_p = c+\epsilon$. If all types $\theta \in \mathcal{E}(1/2)$ choose market s and submit the pooling bid, then the probability of winning conditional on $P = b_p$ converges to κ_s/g and $\kappa_s\pi/g(1-\pi)$, in states 1 and 0, respectively. At the limit, the payoff of $\theta \in \mathcal{E}(1/2)$ submitting the pooling bid is given by $\Pr(V = 1|\theta)(1-b_p)\lim\Pr(b_p\, win|V=1) - \Pr(V=0|\theta)b_p\lim\Pr(b_p\, win|V=1) = \frac{\pi\kappa_s}{g}(1-2c-2\epsilon)$. At the limit, the payoff of $\theta \in \mathcal{E}(1/2)$ submitting a bid greater than the pooling bid is $(1-b_p)\pi - (1-\pi)b_p = \pi - b_p < 0$. Therefore, at the limit, each $\theta \in \mathcal{E}(1/2)$ prefers submitting the pooling bid instead of bidding slightly above the pooling bid and winning with probability one whenever the price is equal to the pooling bid. The fact that each $\theta \in \mathcal{E}(1/2)$ strictly prefers the pooling bid to submitting a bid greater than the pooling bid at the limit implies that these types also prefer the pooling for sufficiently large n. Also, if a type $\theta \in \mathcal{E}(1/2)$ submits a bid less than the pooling bid, then they never win a object at the limit. And, since their payoff at pooling is positive, they prefer the pooling bid to undercutting the pooling bid. Types $\theta \in \mathcal{E}(1)$ opt for the outside option because $b_p > c$ and all types $\theta \in \mathcal{E}(0)$ submit a bid equal to zero. $\square$

B.3. Proof of Proposition 4.2. The assertions concerning market r follow from Lemma A.13.

Claim B.1. If $c < \bar{c}$ or if $c > 1/2$, then there is no pooling by pivotal types. Therefore $\lim\sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| < \infty$.

Proof. Suppose instead that $\lim\sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| \to \infty$. We will argue below that if $c < \bar{c}$ or if $c > 1/2$, then pooling by pivotal types cannot be sustained. However, if there is no pooling by pivotal types and if $\lim\sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1)| \to \infty$, then information is aggregated. Therefore, once we conclude that pooling by pivotal types cannot be sustained, this conclusion and Theorem 3.1 together imply that $\lim\sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1)| < \infty$.

Assume pooling by pivotal types, $\lim\Pr(P^n \leq b_p^n|V=1) = 1$ and $\lim\Pr(P^n < b_p^n|V = 0) = 0$. Let $\liminf b_p^n = b_p$. Equation A.1 shows

$$b_p/(1-b_p) \geq \lim \frac{\Pr(b_p^n\, lose, P_s^n = b_p^n|V=1)}{\Pr(b_p^n\, lose, P_s^n = b_p^n|V=0)} l(\theta^n) \tag{B.1}$$

for each θ^n that submits the pooling bid b_p^n along any subsequence where the limits exist. We now argue that $b_p \lim \Pr(b_p^n\, win, P_s^n = b_p^n | V=0) \leq c$, $(1-b_p)\lim \Pr(b_p^n\, win, P_s^n = b_p^n | V=1) \leq 1-c$ and therefore

$$b_p/(1-b_p) \leq \lim \frac{\Pr(b_p^n\, win, P_s^n = b_p^n | V=1)}{\Pr(b_p^n\, win, P_s^n = b_p^n | V=0)} c/(1-c). \tag{B.2}$$

Suppose that $b_p \lim \Pr(b_p^n\, wins, P_s^n = b_p^n | V=0) > c$, then any $\theta > \underline{\theta}_p$, and therefore any type $\theta > \theta_s(1)$, would select market s by Lemma A.10. If so, then MLRP implies that $F_s(\theta_s(1)|1) < F(\theta_s(1)|0)$, which leads to a contradiction because in this case information would be aggregated by Lemma A.9. Suppose that $(1-b_p)\lim \Pr(b_p^n\, win, P_s^n = b_p^n | V = 1) > 1-c$. If this inequality were true, then any type would prefer the pooling bid to market r because state-by-state profits are higher at the pooling bid, which again leads to a contradiction. Hence, inequalities B.1 and B.2 are satisfied if there is pooling by pivotal types.

We first show that pooling by pivotal types is not possible if $c < \bar{c}$. We show that inequalities B.1 and B.2 together imply that $c/(1-c) \geq \bar{c}/(1-\bar{c})$. The fact that $\lim \Pr(P_s^n \leq b_p^n | V = 1) = 1$ implies that $b^n(\theta_s^n(1)) = b_p^n$ for sufficiently large n, which establishes inequality B.1 for $\theta^n = \theta_s^n(1)$. Therefore, inequalities B.1 and B.2 imply

$$c/(1-c) \geq \lim \frac{\Pr(b_p^n\, lose, P_s^n = b_p^n | V=1)}{\Pr(b_p^n\, win, P_s^n = b_p^n | V=1)} \frac{\Pr(b_p^n\, win, P_s^n = b_p^n | V=0)}{\Pr(b_p^n\, lose, P_s^n = b_p^n | V=0)} l(\theta_s^n(1)).$$

There are two possibilities: either $F_s(\underline{\theta}_p|1) = 0$, or $F_s(\underline{\theta}_p|1) > 0$.

Case 1: Suppose that $F_s(\underline{\theta}_p|1) = 0$. In this case, Lemma A.2 and $F_s(\underline{\theta}_p|1) = 0$ together imply that

$$\begin{aligned} \lim \frac{\Pr(b_p^n\, lose, P_s^n = b_p^n | V=1)}{\Pr(b_p^n\, win, P_s^n = b_p^n | V=1)} &= \frac{F_s(1|1) - \kappa_s}{\kappa_s - (F_s(1|1) - F_s(\theta_p|1))} \\ \lim \frac{\Pr(b_p^n\, win, P_s^n = b_p^n | V=0)}{\Pr(b_p^n\, lose, P_s^n = b_p^n | V=0)} &= \frac{\kappa_s - (F_s(1|0) - F_s(\theta_p|0))}{F_s(1|0) - \kappa_s}. \end{aligned}$$

Hence,

$$c/(1-c) \geq \frac{F_s(1|1) - \kappa_s}{F_s(1|0) - \kappa_s} \frac{\kappa_s - (F_s(1|0) - F_s(\theta_p|0))}{\kappa_s - (F_s(1|1) - F_s(\theta_p|1))} \lim l(\theta_s^n(1)) \geq \frac{(1-\kappa_r-\kappa_s)^2}{1-\kappa_s} \geq \bar{c}/(1-\bar{c}),$$

where the second inequality is satisfied because $F_r(1|1) \leq \kappa_r$, $\frac{\kappa_s - (F_s(1|0) - F_s(\theta_p|0))}{\kappa_s - (F_s(1|1) - F_s(\theta_p|1))} \geq 1$, and

$\lim l(\theta_s^n(1)) \geq 1-\kappa_r-\kappa_s$. To see $\lim l(\theta_s^n(1)) \geq 1-\kappa_r-\kappa_s$, note that

$$\lim l(\theta_s^n(1)) \geq \frac{F(\theta_s(1)|1)}{F(\theta_s(1)|0)} \geq F(\theta_s(1)|1) \geq F_s(\theta_s(1)|1) \geq F_s(1|1)-\kappa_s \geq 1-\kappa_r-\kappa_s,$$

We now argue that $\frac{(F_s(1|0)-F_s(\theta_p|0))}{(F_s(1|1)-F_s(\theta_p|1))} \leq 1$, which implies $\frac{\kappa_s-(F_s(1|0)-F_s(\theta_p|0))}{\kappa_s-(F_s(1|1)-F_s(\theta_p|1))} \geq 1$. Inequality B.1, Lemma A.2, and $F_s(\underline{\theta}_p|1) = 0$ together imply that $b_p/(1-b_p) \geq \lim \Pr(b_p^n \; lose, P_s^n = b_p^n|V=1)l(\theta_s^n(1)) \geq \frac{(1-\kappa_r-\kappa_s)^2}{1-\kappa_r} \geq \frac{\bar{c}}{1-\bar{c}}$. Therefore, if $\frac{c}{1-c} < \frac{\bar{c}}{1-\bar{c}}$, then $b_p > c$, and any type $\theta > \theta_p$ would select market s by Lemma A.10. This implies that $\frac{(F_s(1|0)-F_s(\theta_p|0))}{(F_s(1|1)-F_s(\theta_p|1))} = \frac{(F(1|0)-F(\theta_p|0))}{(F(1|1)-F(\theta_p|1))} < 1$, as claimed. Thus pooling cannot be sustained if $c/(1-c) < \bar{c}/(1-\bar{c})$ and $F_s(\underline{\theta}_p|1) = 0$.

Case 2: Suppose instead that $F_s(\underline{\theta}_p|1) > 0$. In this case, Lemma A.2 and $F_s(\underline{\theta}_p|1) > 0$ together imply that

$$\begin{aligned}
\lim \frac{\Pr(b_p^n \, lose, P_s^n = b_p^n|V=1)}{\Pr(b_p^n \, win, P_s^n = b_p^n|V=1)} &= \frac{F_s(\theta_s(1)|1)-F_s(\underline{\theta}_p|1)}{\kappa_s-(F_s(1|1)-F_s(\theta_p|1))} \\
\lim \frac{\Pr(b_p^n \, win, P_s^n = b_p^n|V=0)}{\Pr(b_p^n \, lose, P_s^n = b_p^n|V=0)} &\geq \frac{\kappa_s-(F_s(1|0)-F_s(\theta_p|0))}{F_s(1|0)-\kappa_s}.
\end{aligned}$$

Hence, $c/(1-c) \geq \frac{F_s(\theta_s(1)|1)-F_s(\underline{\theta}_p|1)}{F_s(1|0)-\kappa_s} \frac{\kappa_s-(F_s(1|0)-F_s(\theta_p|0))}{\kappa_s-(F_s(1|1)-F_s(\theta_p|1))} \lim l(\theta_s^n(1)) \geq x(1-\kappa_r-\kappa_s)/(1-\kappa_s)$, where $x = F_s(\theta_s(1)|1)-F_s(\underline{\theta}_p|1)$. In establishing the final inequality, we used the fact that $\frac{\kappa_s-(F_s(1|0)-F_s(\theta_p|0))}{\kappa_s-(F_s(1|1)-F_s(\theta_p|1))} \geq 1$. We provide an argument for this further below.

Any type such that $F_s(\theta^n|1) = F_s(\underline{\theta}_p^n|1)-\epsilon$ for $\epsilon > 0$, who bids in market s, has a payoff equal to zero. Any such type can submit the pooling bid. Therefore,

$$\lim((1-b_p^n)\Pr(b_p^n \, win, P_s^n = b_p^n|1)\Pr(1|\theta^n) - b_p^n \Pr(b_p^n \, win, P_s^n = b_p^n|0)\Pr(0|\theta^n)) \leq 0$$

Rearranging, we conclude $b_p/(1-b_p) \geq \lim l(b_p^n \, win, P_s^n = b_p^n)l(\theta^n)$. Note that inequality B.2 implies that $b_p/(1-b_p) \leq \lim l(b_p^n \; win, P_s^n = b_p^n)c/(1-c)$. Combining these two inequalities, observing that $\lim l(\theta^n) \geq F_s\left(\underline{\theta}_p^n|1\right) - \epsilon \geq 1-\kappa_s-\kappa_r-x-\epsilon$ and using the fact that ϵ is arbitrary we conclude $c/(1-c) \geq 1-\kappa_s-\kappa_r-x$. Therefore, $c/(1-c) \geq \max\{1-\kappa_s-\kappa_r-x, \frac{x(1-\kappa_r-\kappa_s)}{1-\kappa_s}\} \geq (1-\kappa_r-\kappa_s)^2/(1-\kappa_s+1-\kappa_s-\kappa_r) \geq \bar{c}/(1-\bar{c})$, where we obtain the lower bound by solving for the value of x that minimizes the expression inside the maximum function.[27]

We now complete the argument by showing that $\frac{\kappa_s-(F_s(1|0)-F_s(\theta_p|0))}{\kappa_s-(F_s(1|1)-F_s(\theta_p|1))} \geq 1$. Any type such that $F_s(\theta^n|1) = F_s(\underline{\theta}_p^n|1)-\epsilon$ for $\epsilon > 0$ who bids in market s has a payoff

[27] Observe that, inside the maximum, we have two linear functions, one increasing and the other decreasing in x. Therefore, this expression in minimized at the value of x where the two expressions are equal.

equal to zero. Any such type can bid $b_p+\epsilon$ and win an object with probability that converges to one. Thus, we must have $\lim(1-b_p^n)l(\theta^n)-b_p^n\le 0$ for any such type. Therefore, $\frac{b_p}{1-b_p}\ge \lim l(\theta^n)\ge F_s(\underline{\theta}_p^n|1)\ge 1-\kappa_s-\kappa_r-x$ because ϵ is arbitrary. Moreover, Inequality B.1, Lemma A.2 and $F_s(\underline{\theta}_p|1)>0$ together imply that $b_p/(1-b_p)\ge \lim \Pr(b_p^n\, lose, P_s^n=b_p^n|V=1)l(\theta_s^n(1))\ge x(1-\kappa_r-\kappa_s)/(1-\kappa_r)$. Therefore, $b_p/(1-b_p)\ge \max\{1-\kappa_s-\kappa_r-x, x(1-\kappa_r-\kappa_s)/(1-\kappa_r)\}\ge(1-\kappa_s-\kappa_r)^2/(1-\kappa_r+1-\kappa_s-\kappa_r)\ge\bar{c}/(1-\bar{c})$, where to obtain the lower bound we solve for x by observing that inside the maximum we have two linear functions of x, one increasing and the other decreasing in x. Therefore, if $\frac{c}{1-c}<\frac{\bar{c}}{1-\bar{c}}$, then $b_p>c$, and any type $\theta>\theta_p$ would select market s by Lemma A.10. This implies that $\frac{(F_s(1|0)-F_s(\theta_p|0))}{(F_s(1|1)-F_s(\theta_p|1))}=\frac{(F(1|0)-F(\theta_p|0))}{(F(1|1)-F(\theta_p|1))}<1$ as claimed.

We now argue that pooling by pivotal types is not possible if $c>1/2$. We will show that $\lim l(b_p^n\ win, P_s^n=b_p^n)l\left(\underline{\theta}_p^n\right)\le 1<(1-b_p)b_p$ which implies that $(1-b_p)\Pr(b_p^n\ win, P_s^n=b_p^n|1)l(\underline{\theta}_p^n)-b_p\Pr(b_p^n\ win, P_s^n=b_p^n|0)\to 0$ along any subsequence where these limits exist. However, then any type sufficiently close to $\underline{\theta}_p$ makes a loss at pooling, which contradicts that there is a pooling interval as claimed.

We now argue that $\lim l(b_p^n\, win, P_s^n=b_p^n)l(\underline{\theta}_p^n)<1$. Lemma A.2 implies that

$$\lim\frac{\Pr(b_p^n\, win, P_s^n=b_p^n|1)}{\Pr(b_p^n\, win, P_s^n=b_p^n|0)}l(\underline{\theta}_p^n)=\frac{\kappa_s-(F_s(1|1)-F_s(\theta_p|1))}{\kappa_s-(F_s(1|0)-F_s(\theta_p|0))}\frac{F_s(\theta_p|0)-F_s(\underline{\theta}_p|0)}{F_s(\theta_p|1)-F_s(\underline{\theta}_p|1)}l(\underline{\theta}_p^n).$$

Below we show $\frac{F_s(1|1)-F_s(\theta_p|1))}{F_s(1|0)-F_s(\theta_p|0)}>1$ and therefore $\frac{\kappa_s-(F_s(1|1)-F_s(\theta_p|1))}{\kappa_s-(F_s(1|0)-F_s(\theta_p|0))}<1$. Also, MLRP implies that $\frac{F_s(\theta_p|0)-F_s(\underline{\theta}_p|0)}{F_s(\theta_p|1)-F_s(\underline{\theta}_p|1)}\lim l(\underline{\theta}_p^n)\le 1$ proving the claim.

We now argue that $\frac{F_s(1|1)-F_s(\theta_p|1))}{F_s(1|0)-F_s(\theta_p|0)}>1$ and therefore $\frac{\kappa_s-(F_s(1|1)-F_s(\theta_p|1))}{\kappa_s-(F_s(1|0)-F_s(\theta_p|0))}<1$. Note that $b_p\ge c>1/2$ because otherwise bidding just above the pooling bid delivers a better payoff than bidding in market r for any type. However, if $b_p>1/2$, then submitting a bid that exceeds b_p delivers a strictly negative payoff for any type with $l(\theta)\le 1$. This is because any type such that $F_s(\theta|1)>F_s(\theta_p|1)$ that bids in market s wins an object with probability one in both states and pays a price that strictly exceeds $1/2$. However, this delivers a negative payoff for any type with $l(\theta)\le 1$. However, if $l(\theta)>1$ for all $\theta>\theta_p$ that bid in market s, then $\frac{F_s(1|1)-F_s(\theta_p|1))}{F_s(1|0)-F_s(\theta_p|0)}>1$. □

Claim B.2. If $c<\bar{c}$, then $\lim\mathbb{E}[P_r^n|V=1]=\lim\mathbb{E}[P_s^n|V=1]$ and $\lim\mathbb{E}[P_r^n|V=0]=\lim\mathbb{E}[P_s^n|V=0]=c$.

Proof. Note that $F_r(1|0)<\kappa_r$ implies that $\lim\mathbb{E}[P_r^n|V=0]=c$. If $c>\lim\mathbb{E}[P_s^n|V=0]$, then $a_r(\theta)=1$ for all $\theta>\hat{\theta}_r$ by Lemma A.11. This implies that $F_s(\theta_s(0)|1)>F_s(\theta_s(1)|1)$ because $\kappa_s>\bar{\kappa}_{en}$ and because $\hat{\theta}_r=\theta_{en}$, which contradicts that the pivotal types are arbitrarily close.

Further below we show any type $\theta > \theta_s(0)$ who submits a bid in market s wins an object with probability one if $V=0$. If $c < \lim\mathbb{E}[P_s^n|V=0]$, then $a_s(\theta)=1$ for all $\theta > \theta_s(0)$ by Lemma A.11. However, this implies that $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$, which again contradicts that the pivotal types are arbitrarily close.

We now show that for any type $\theta' > \theta_s(0)$, $\lim\Pr(b_s^n(\theta')\, win, P_s^n \leq b_s^n(\theta')|V=0)=1$. Suppose that $\theta' > \theta_s(0)$ and $\lim\Pr(b_s^n(\theta')\, win, p_s^n \leq b_s^n(\theta')|V=0) < 1$. The LLN implies that $\lim\Pr(P_s^n \leq b_s^n(\theta_s^n(0))|V=0)=1$. Moreover, monotonicity of bidding implies that $b_s^n(\theta') \geq b_s^n(\theta_s^n(0))$. Therefore, it must be the case that $b_s^n(\theta') = b_s^n(\theta_s^n(0)) = b_p^n$ for all sufficiently large n. Hence there must be a sequence of pooling regions $(\underline{\theta}_p^n, \theta_p^n)$ such that $\underline{\theta}_p \leq \theta_s(0) = \theta_s(1) < \theta' \leq \theta_p$. Our assumption that $\lim\Pr(b_s^n(\theta')\, win, p_s^n \leq b_s^n(\theta')|V=0) < 1$ implies that $\underline{\theta}_p < \theta_s(0)$ because otherwise all bidders in the pooling region would win a good with probability one by Lemma A.2. However, we showed that if $c < \bar{c}$,then such a pool is not possible in Claim B.1.

We complete the proof by showing that $\lim\mathbb{E}[P_s^n|V=0] = \lim\mathbb{E}[P_r^n|V=0] = c$ implies $\lim\mathbb{E}[P_r^n|V=1] = \lim\mathbb{E}[P_s^n|V=1]$. Note that any type $\theta > \hat{\theta}_r$ who bids in market r wins an object with probability converging to one in both states. If, however, $\lim\mathbb{E}[P_r^n|V=1] > \lim\mathbb{E}[P_s^n|V=1]$, then any such type would prefer to submit a bid equal to one in market s. Similarly, we argued above that any type $\theta' > \theta_s(0)$ wins an object with probability one in both states. However, if $\lim\mathbb{E}[P_r^n|V=1] < \lim\mathbb{E}[P_s^n|V=1]$, then any such type would prefer to submit a bid equal to one in market r. $\square$