Market Selection and the Information Content of Prices*

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Abstract

We study information aggregation when n bidders choose, based on their private information, between two concurrent common-value auctions. There are k_s identical objects on sale through a uniform price auction in market s and there are an additionally k_r objects on auction in market r, which is identical to market s except for a positive reserve price. The reserve price in market r implies that information is not aggregated in this market. Moreover, if the object-to-bidder ratio in market s exceeds a certain cutoff, then information is not aggregated in market s either. Conversely, if the object-to-bidder ratio is less than this cutoff, then information is aggregated in market s as the market grows arbitrarily large. Our results demonstrate how frictions in one market can disrupt information aggregation in a linked, frictionless market because of the pattern of market selection by imperfectly informed bidders.

Keywords: Auctions, large markets, information aggregation.

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1. Introduction

Consider a market where k_s identical common-value objects of unknown value are sold to n bidders, each with unit demand. The sale is conducted through a sealed-bid auction where each of the highest k_s bidders receives an object and pays a uniform price equal to the highest losing bid. Each object's common value (V) is equal to one in the good state and zero in the bad state. In such an auction, if each bidder has an independent signal about the unknown value of the object, then the auction's equilibrium price converges to the object's true value as the number of objects and the number of bidders grow arbitrarily large (see Pesendorfer and Swinkels (1997)). Therefore, the auction price reveals the unknown value of the object and thus aggregates all relevant information dispersedly held by the bidders.

Most previous work on auctions takes the distribution of types that bid in the auction as exogenously given.[1] Yet, in many instances, bidders strategically decide whether to trade in a particular market after weighing their alternatives. In other words, the bidder distribution is endogenously determined jointly by the set of available alternatives and the bidders' expectations about the relative attractiveness of these alternatives. Our focus in this paper is an environment where bidders choose, based on their private information, between the auction (market s) and an outside option (market r). This framework allows us to highlight the interplay between self-selection into an auction, bidding behavior in the auction, and the information content of prices.

Market r, which serves as the outside option for market s, is a uniform-price auction with a reserve price $c > 0$ where there are an additional $n\kappa_r = k_r$ units of the same object for sale.[2] If the object-to-bidder ratio in market r is sufficiently large, then each bidder can purchase an object at a fixed price equal to $c > 0$. In this case, the payoff from choosing the outside option is *exogenously* determined by the reserve price c. Otherwise, the attractiveness of the outside option is *endogenously* determined by the bidders that select market r together with the reserve price c.

Our main result identifies when frictions in market r, resulting from the positive reserve price, disrupts information aggregation also in the frictionless market s. In particular, we show that there is no symmetric equilibrium that aggregates information in either market if the object-to-bidder ratio in market s exceeds a

[1] Papers by Lauermann and Wolinsky (2017) and Murto and Valimaki (2014) are notable exceptions.

[2] The reserve price has various interpretations: (1) It is a reserve price set by a single auctioneer, (2) The auction is comprised of k_r nonstrategic sellers and the reservation value (or the cost) for these sellers is equal to c; (3) It is a government/regulator imposed minimum price.

certain cutoff $\bar{\kappa}$. This cutoff depends on the reserve price, the signal structure, and the object-to-bidder ratio in market r. If, on the other hand, the object-to-bidder ratio in market s is less than $\bar{\kappa}$, then information is aggregated in market s. Importantly, our result implies that information aggregation can fail in both markets under imperfect information even in circumstances where information is aggregated in both markets under complete information. We provide intuition for these findings using an illustrative example further below.

Previous work on information aggregation mainly focused on homogeneous (or highly correlated) objects that trade in a single centralized, frictionless auction market. However, such a centralized market is an exception rather than the rule. Fragmentation, the disperse trading of the same security in multiple markets, is commonplace: Many stocks listed on the New York Stock Exchange trade concurrently on regional exchanges (see Hasbrouck (1995)). Investors, who participate in a primary treasury bond auction, could purchase a bond with similar cashflow characteristics from the secondary market. Labor markets are linked but also segmented according to industry, geography, and skill. Buyers in the market for aluminum or steel can choose between the London Metal Exchange or the New York Mercantile Exchange. Such fragmented markets and exchanges also differ in structure, rules and regulations. In particular, markets are heterogeneous in terms of the frictions that participants face. The results that we present in this paper suggest that selection into markets can have important implications for the information content of prices, especially when individuals choose between markets that differ in terms of institutional detail and therefore frictions. In particular, we demonstrate how frictions can disrupt information aggregation not only in the market with frictions but also in frictionless, substitute markets.

1.1. An Illustrative Example. Recall that bidders choose, based on their private information, between market s where there are $n\kappa_s$ objects on auction and market r where there are an additional $n\kappa_r$ objects on auction. There is a positive reserve price $c > 0$ in market r while there is no reserve price in market s. For this example, assume that $\kappa_r + \kappa_s < 1$ and further suppose that each bidder receives a private signal that perfectly reveals the value of the object with probability $1 - g$ and receives an uninformative signal with the remaining probability $g \in [0, 1]$. A bidder who receives the uninformative signal believes that $V = 1$ with probability $1/2$ while a bidder who receives the perfectly revealing signal knows the object's true value.

As a first benchmark suppose that $\kappa_r = 0$, i.e., suppose that there is only one active market. In this case, it is innocuous to assume that all bidders participate in market s because a non-participating bidder's payoff is equal to zero in both states. However, if all bidders participate in market s, then Pesendorfer and Swinkels (1997)'s analysis implies that the auction price in market s converges to 1 and 0 in state $V = 1$ and $V = 0$, respectively, as the number of bidders n and the number of object $n\kappa_s$ grow arbitrarily large for any $g < 1$. In other words, if $\kappa_r = 0$, then information is aggregated because the auction price in market s converges to the object's value in each state.

As a second benchmark assume that $\kappa_r > 0$ and suppose that all bidders receive perfectly informative signals ($g = 0$). In this benchmark, there is a unique equilibrium for each n and information is again aggregated. In state $V = 0$ all bidders bid zero in auction s because there is a positive reserve price in market r. Therefore, the price in state $V = 0$ is equal to zero and c in markets s and r, respectively.[3] In state $V = 1$, the bidders randomize between the two auctions and bid one in the auction that they choose.[4] Since the bidders randomize, they are indifferent between the two markets in equilibrium. Moreover, the facts that all bidders bid one and $\kappa_r + \kappa_s < 1$ together imply that the price in one of the two markets must converge to one. Since the bidders are indifferent between the two markets, the price in state $V = 1$ must converge to one in both markets. Therefore, the auction price in market s converges to value and perfectly reveals the state.

In contrast to the two benchmarks, we will now argue that price cannot converge to value in market s if there are sufficiently many uninformed bidders.[5] For this argument, we will assume that $1 - g < \kappa_r$ and $c > 1/2$. On the way to a contradiction, assume that price converges to value in auction s. No uninformed bidder and no bidder who knows that the state is $V = 0$ would bid in market r in equilibrium because the price in this market is at least $c > 1/2$ in both states. Consider a bidder who knows that the state is $V = 1$. This bidder's payoff from participating in auction s converges to zero because the auction price converges to one in state $V = 1$ by our initial assumption. The price in market r converges to c in both states because $1 - g < \kappa_r$ and because only the informed bidders select market r. Therefore, any informed bidder will opt for market r in state $V = 1$ for

[3] In the auction that we study, if there are fewer bidders than objects, then the price is equal to the reserve price.

[4] Bidders bid their value in the auctions since the auctions are $k_s + 1$ (or $k_r + 1$) price auctions.

[5] A third benchmark that comes to mind is one where the reserve price c in market r is also equal to zero. In this case, price converges to value in both markets along every equilibrium sequence.

sufficiently large n. However, if no bidder other than the uninformed bidders submit nontrivial bids that exceed zero in market s, then all the uninformed bidders would bid $1/2$, i.e., their valuation for the object. Thus, the price cannot converge to one in state $V = 1$, contradicting our initial assumption.

This example highlights the main tension between type dependent market selection and information aggregation. In order for information to be aggregated in market s, informed bidders must choose this market in both states. However, if information is aggregated, then no informed bidder would choose market s in state $V = 1$ because they can obtain an object for a price equal to c in market r. We further discuss this example in section 4.

1.2. Relation to the Literature. We make two main contributions to the literature on information aggregation in multi-object common-value auctions. (1) We are the first to study bidding behavior in a multi-object common-value auction where bidders have outside options and the distribution of types is endogenously determined. (2) In this context, we highlight a new mechanism, based on self-selection, that can lead to the failure of information aggregation.

The model that we study is closest to Pesendorfer and Swinkels (1997).[6] Their paper argued that prices converge to the true value of a common-value object in all symmetric equilibria if and only if both the number of objects and the number of bidders who are not allocated an object grow without bound (double-largeness). In contrast, we show that information aggregation can fail if bidders have access to an outside option even when the double-largeness condition is satisfied.

Our paper is also related to recent work on single-unit common-value auctions by Lauermann and Wolinsky (2017) and Murto and Valimaki (2014). The novel feature of Lauermann and Wolinsky (2017)'s model is that the auctioneer knows the value of the object but must solicit bidders for the auction, and soliciting bidders is costly. Therefore, the number of bidders in the auction is endogenously determined by the auctioneer. Our paper differs from Lauermann and Wolinsky (2017) because: (1) We study a multi-unit multi-market auction, while they study a single-object single-market auction, and Pesendorfer and Swinkels (1997)'s analysis implies that the information aggregation properties of a multi-unit auctions

[6]There is extensive work on information aggregation by prices in various contexts. For example, see Wilson (1977) for common-value, uniform-price auctions with one object for sale; Pesendorfer and Swinkels (2000) for mixed private, common-value auctions; Reny and Perry (2006) and Cripps and Swinkels (2006) for large double auctions; Vives (2011) and Rostek and Weretka (2012) for markets for divisible objects; and Wolinsky (1990), Golosov et al. (2014), Ostrovsky (2012), Lauermann and Wolinsky (2015), and Lambert et al. (2018) for search markets and markets with dynamic trading.

differ substantially from the information aggregation properties of an auction with a single object. (2) In our model the distribution of types is determined by the participation decision of the bidders, while in their paper the auctioneer's solicitation strategy determines the number of bidders. This implies that in our model participation decisions are type dependent, while in theirs they are type independent but state dependent. In Murto and Valimaki (2014), potential bidders must pay a cost to participate in the auction. This creates type-dependent participation, as in our model. However, in contrast to this paper, they study a single-object, single-market auction and their emphasis is on characterizing equilibria rather than information aggregation.

Lauermann and Wolinsky (2017) and Atakan and Ekmekci (2014) also present models where information aggregation fails in a large common-value auction. In both of these papers, information aggregation fails because there is an atom in the bid distribution and the auction price is equal to this atom with positive probability in both states of the world. In this paper, information aggregation can fail even when there are no atoms in the bid distribution. For instance, in the illustrative example information aggregation fails in market s because the same set of types determine the price and the limit-price distribution is continuous, atomless, and increasing over the unit interval, in both states.

2. Preliminaries

We study an auction where n bidders choose between three mutually exclusive alternatives: 1) A bidder can bid in market s; 2) She can bid in market r; or 3) She can choose neither and receive a payoff equal to zero. A bidder does not observe anything beyond her private signal when making this choice.

Market s is a common-value, sealed-bid, uniform-price auction for $\lceil \kappa_s n \rceil = k_s$ identical objects where $\kappa_s \in (0,1)$ is the object-to-bidder ratio.[7] There are $\lceil n\kappa_r \rceil = k_r$ additional objects on auction in market r and the auction format in market r is identical to market s except for a reserve price $c \in (0,1)$. The price in market s is equal to the $k_s + 1$st highest bid in market s (the highest losing bid) if there are more bidders than objects and equal to zero, otherwise. The price in market r is equal to the maximum of c and the highest losing bid in market r if there are more bidders than objects and equal to c, otherwise. Ties are broken uniformly and randomly.

Each bidder has unit demand and puts value V on a single object, and value 0 on any further objects. The k_m highest bidders in auction $m \in \{r, s\}$ are allocated

[7] The smallest integer not less than x is denoted by $\lceil x \rceil$.

objects. Thus, a bidder who is allocated an object at price P enjoys utility $V - P$ while a bidder who fails to win an object receives a payoff equal to zero.

The common value V (or the state of the world) is a random variable with typical realizations $v \in \{0, 1\}$. The common prior is equal to $1/2$.[8] Before selecting a market, each bidder receives a signal $\theta \in [0, 1]$ according to a continuous, increasing cumulative distribution function $F(\theta|v)$ that admits a density function $f(\theta|v)$, $v = 0, 1$.[9] Conditional on V, the signals are identically and independently distributed. Given that there are two states of the world, the signals satisfy the monotone likelihood ratio property **(MLRP)** possibly after a reordering. In other words, the likelihood ratio $l(\theta) := f(\theta|1) / f(\theta|0)$, is a nondecreasing function of θ. Throughout the paper, we further assume that (1) **there are no uninformative signals**, that is, $F(\{\theta : l(\theta) = 1\}) = 0$; and (2) **signals contain bounded information**, i.e., there is a constant $\eta > 0$ such that $\eta < l(\theta) < \frac{1}{\eta}$ for all $\theta \in [0, 1]$. The first assumption states that the mass of signals that contain no information is equal to zero. This is a strengthening of MLRP, but it is weaker than assuming strict MLRP. The second assumption is a technical condition that is also maintained by Pesendorfer and Swinkels (1997). These assumptions significantly simplify the statements and proofs of our results. However, neither of these two assumptions is needed to show that information aggregation fails under the other assumptions outlined in the paper. In fact, in the illustrative example neither assumption is satisfied but all of our results nevertheless hold.

2.1. Strategies and Equilibrium. We represent bidder behavior by a distributional strategy H, which is a measure over $[0, 1] \times \{s, r, \mathit{neither}\} \times [0, \infty)$. We focus on the symmetric Nash equilibria of the game Γ in which all players use the same distributional strategy H and we refer to a symmetric strategy profile simply by the strategy H. We ignore, without loss of generality, the option of choosing "neither" because this option is never chosen by a positive measure of types in any symmetric equilibrium.[10]

For a given strategy H, define the **measure of types in auction** s by $F_s^H(\theta) := H([0, \theta] \times \{s\} \times [0, \infty))$ and define the **selection function** $a^H : [0, 1] \to [0, 1]$ as the function such that $F_s^H(\theta) = \int_0^\theta a^H(\theta) dF(\theta)$. Intuitively, $a^H(\theta)$ is the probability that type θ bids in auction s. Also, $F_s^H(\theta|v) := \int_0^\theta a^H(\theta) dF(\theta|v)$ is

[8] We focus on a uniform prior for expositional simplicity only.

[9] For any half-open interval $(\theta', \theta'']$, we use $F((\theta', \theta'']|v) := F(\theta''|v) - F(\theta'|v)$.

[10] If a positive mass of types were to choose "neither" in a symmetric equilibrium, then any bidder who submits a bid equal to zero in auction s would win an object with strictly positive probability in state $V = 1$. Thus, all types who choose "neither" and receive a payoff equal to zero would rather bid zero in the auction and receive a strictly positive expected payoff.

the measure of types that bid in market s conditional on $V = v$ and $\bar{F}_s^H(\theta|v) := F_s^H(1|v) - F_s^H(\theta|v)$. The kth highest type that bids in auction s is denoted by $Y_s^n(k)$, and we set $Y_s^n(k)$ equal to zero if there are fewer than k bidders in the auction.

The following lemma, which follows from Pesendorfer and Swinkels (1997, Lemmata 3-7), allows us to work exclusively with a pure and nondecreasing bidding strategy, i.e., a function $b : [0, 1] \to [0, \infty)$ such that $H(\{\theta, s, b(\theta)\}_{\theta \in [0,1]}) = F_s^H(1)$. Moreover, if the bidding function is increasing over an interval of types, then any type θ in this interval bids her value conditional on being the pivotal bidder in the auction.

Lemma 2.1. *Any equilibrium H can be represented by a nondecreasing bidding function b^H. Moreover, if $b^H(\theta)$ is increasing over an interval (θ', θ''), then $b^H(\theta) = \mathbb{E}[V|Y_s^{n-1}(k_s) = \theta, \theta]$ for almost every $\theta \in (\theta', \theta'')$.*

Below we define a certain type $\theta_s^H(v)$ for each state v such that the expected number of bids above this type's bid in state v is exactly equal to the number of goods in market s. We refer to $\theta_s^H(v)$ as the **pivotal type in state** v because the types that determine the auction price are concentrated around $\theta_s^H(v)$ in a large market by the law of large numbers (LLN).

Definition 2.1 (Pivotal types)**.** For any strategy H, the **pivotal type** in state v is $\theta_s^H(v) := \max\{\theta : \bar{F}_s^H(\theta|v) = \kappa_s\}$, and $\theta_s^H(v) := 0$ if the set is empty.[11]

For any sequence of strategies $\{H^n\}$, we will denote each $\theta_s^{H^n}(v)$ simply by $\theta_s^n(v)$, and we let $\theta_s(v) = \lim_n \theta_s^n(v)$ and $F_s(\theta|v) = \lim F_s^n(\theta|v)$ whenever such limits exist.

2.2. Definition of Information Aggregation. We study a sequence of strategies $\mathbf{H} = \{H^n\}_{n=1}^{\infty}$ for a sequence of auctions Γ^n where the n^{th} auction has n bidders. We assume that the parameters of the auctions are constant along the sequence and satisfy all the assumptions that we make.

Suppose that the number of bidders n is large. In this case, the LLN implies that observing the signals of all n bidders conveys precise information about the state of the world. A strategy H^n determines an auction price P^n given any realization of signals. We say that information is aggregated in the auction if this

[11]The equation $\bar{F}_s^H(\theta|v) = \kappa_s$ can have multiple solutions if F_s^H is flat over a range of θ. However, the function $\bar{F}_s^H(\theta|v)$ is continuous because it is absolutely continuous with respect to $\bar{F}(\theta|v)$. Hence, the set $\{\theta : \bar{F}_s^H(\theta|v) = \kappa_s\} \subset [0, 1]$ is compact and has a unique maximal element if it is nonempty.

price also conveys precise information about the state of the world. Specifically, (i) if the likelihood ratio $l(P^n = p) := \frac{\Pr(V=1|P^n=p)}{\Pr(V=0|P^n=p)}$ is close to zero (i.e., if it is arbitrarily more probable that we observe such a price p when $V = 0$), then an outsider who observes price p learns that the state is $V = 0$. Alternatively, (ii) if the likelihood ratio $l(P^n = p)$ is arbitrarily large, then an outsider who observes price p learns that the state is $V = 1$. If the probability that we observe a price that satisfies either (i) or (ii) is arbitrarily close to one, then we say that the equilibrium sequence aggregates information. Our formal definition of information aggregation is given below:

Definition 2.2. (Kremer (2002) and Atakan and Ekmekci (2014)) A sequence of strategies $\mathbf{H}$ aggregates information if the random variables $l(P^n = p)$ and $1/l(P^n = p)$ converge in probability to zero in state 0 and state 1, respectively.

We now derive conditions that are necessary and sufficient for information aggregation. Information aggregation fails if the supports of the limit price distributions are the same in the two states. The following definition captures such failures using the mass that separates the pivotal types.

Definition 2.3. The pivotal types are **distinct** along a sequence $\mathbf{H}$ if $\lim_n \sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| = \infty$ and the pivotal types are **arbitrarily close** along a sequence $\mathbf{H}$ if $\liminf_n \sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| < \infty$.

Distinct pivotal types is a necessary condition for information aggregation. To see why, recall that the random variable $Y_s^n(k_s+1)$ denotes the k_s+1st highest type that bids in the auction. The auction clears at the bid of this type because bidding is monotone (Lemma 2.1). For large n, the distribution of $Y_s^n(k_s+1)$ in state $V = v$ puts most of the mass within finitely many standard deviations of the pivotal type in state $V = v$ and the standard deviation is approximated by $\sqrt{\kappa_s(1-\kappa_s)/n}$. If the pivotal types are arbitrarily close, i.e., if the pivotal types are separated by finitely many standard deviations, then the same set of types determine the price and the supports of the limit price distributions are the same in the two states. Therefore, information cannot be aggregated.

Information aggregation also fails if the limit price distribution features an atom that occurs with positive probability in both states. We term such a failure pooling by pivotal types and formally define it below.

Definition 2.4. There is **pooling by pivotal types** along a sequence $\mathbf{H}$ if there is a subsequence of pooling bids $\{b_p^{n_k}\}$ such that $\lim_k \Pr(P^{n_k} = b_p^{n_k}|V = v) > 0$ for $v = 0, 1$. Otherwise, there is **no pooling by pivotal types**.

No pooling by pivotal types is also a necessary condition for information aggregation because if it does not hold, then the limit price distribution features an atom that occurs with positive probability in both states. In the following lemma, we further show that these two necessary conditions are also sufficient for information aggregation.

Lemma 2.2. *An equilibrium sequence aggregates information if and only if the pivotal types are distinct and there is no pooling by pivotal types.*

A sketch of the argument for sufficiency is as follows: Pick any type θ that is within finitely many standard deviations of the pivotal type in state $V = 1$ and note that the auction can clear only at the bids of such types in state $V = 1$. Distinctness of the pivotal types implies that type θ is infinitely many standard deviations away from the pivotal type in state $V = 0$. Therefore, if type θ does not bid in an atom, then an outside observer, who observes a price equal to type θ's bid, is arbitrarily certain that the state is $V = 1$. On the other hand, suppose that θ bids in an atom, i.e., suppose that the price is equal to θ's bid with positive probability in state $V = 1$. In this case, the probability that the price is equal to θ's bid in state $V = 0$ is equal to zero because there is no pooling by pivotal types. Once again, an outside observer, who observes a price equal to θ's bid, is arbitrarily certain that the state is $V = 1$.

3. Information Aggregation

This section's main theorem shows that information is not aggregated in market s along any equilibrium sequence if the object-to-bidder ratio in market s exceeds a certain cutoff $\bar{\kappa}$ (described further below). Conversely, if the object-to-bidder ratio in market s is less than $\bar{\kappa}$, then information is aggregated in market s along every equilibrium sequence.

In order to state our main theorem, we first define the cutoff $\bar{\kappa}$. Let $\theta_r^F(1)$ denote the pivotal type in market r in state $V = 1$ if all types were to bid in auction r, that is, $\theta_r^F(1)$ is the unique type that satisfies the equality $1 - F(\theta_r^F(1)|1) = \kappa_r$. For a given type $\theta' < 1$, let $\theta^*(\theta')$ denote the unique type $\theta < \theta'$ such that $F([\theta, \theta']|0) = F([\theta, \theta']|1)$, and let $\theta^*(\theta') = \theta'$ if there is no such $\theta < \theta'$.[12] For some intuition, suppose that types $\theta > \theta'$ opt for market r, while types $\theta \leq \theta'$ bid in auction s. In this case, $\theta^*(\theta')$ is defined as the type such that the expected number of bidders who bid in auction s with signals that exceed $\theta^*(\theta')$ is the same in both

[12] If $l(\theta') > 1$, then there is a unique type $\theta < \theta'$ such that $F([\theta, \theta']|0) = F([\theta, \theta']|1)$. Otherwise, there is no such type and $\theta^*(\theta') = \theta'$.

states. The implicit function theorem and MLRP together imply that $\theta^*(\theta')$ is a decreasing function of θ'.[13]

Definition 3.1. Let $\theta_{en} := \max\{\theta_r^F(1), \inf\{\theta : \Pr(V = 1|\theta) > c\}\}$ and $\theta_{en} := 1$ if the set over which the infimum is taken is empty. Define $\bar{\kappa} := F([\theta^*(\theta_{en}), \theta_{en}]|0) = F([\theta^*(\theta_{en}), \theta_{en}]|1)$.

To better understand the definition of $\bar{\kappa}$, suppose that all types greater than θ_{en} select market r while all types smaller than θ_{en} bid in market s. The cutoff $\bar{\kappa}$ is defined so that if the object-to-bidder ratio in market s is equal to $\bar{\kappa}$, then the pivotal type in market s is equal to $\theta^*(\theta_{en})$ in both of the states. Turning next to the definition of θ_{en}, further suppose that any type that chooses market r bids according to an increasing bidding function. Type θ_{en} is defined as the smallest type that can make positive profits in an arbitrarily large market r. To see why the definition captures this property, note that θ_{en} must be at least as large as $\theta_r^F(1)$ because only those types greater than $\theta_r^F(1)$ can actually win an object in the auction in state $V = 1$. Furthermore, any type $\theta > \theta_r^F(1)$ can make a profit in market r only if $\Pr(V = 1|\theta) > c$ because any such type will win an object with probability one in both states and will pay a price which is at least c. Also, see Figure 3.1 for a graphical depiction of $\bar{\kappa}$.

The main implication of Definition 3.1 is as follows: if the object-to-bidder ratio in market s exceeds $\bar{\kappa}$, then the pivotal type in state 0 exceeds the pivotal type in state 1 whenever all types that value market r select market r. Such an ordering of pivotal types is ruled out by MLRP if all types were to bid in market s. However, if types that exceed $\theta_{en} < 1$ choose market r, then the measure of types that bid in market s is smaller in state 1 than in state 0 as a consequence of MLRP. This implies that $\bar{\kappa}$ is less than one. Therefore, there is an open interval $(\bar{\kappa}, 1)$ such that whenever the object-to-bidder ratio is in this interval, the order of the pivotal types is reversed. The converse is also true, that is, if the object-to-bidder ratio in auction s is less than $\bar{\kappa}$, then the pivotal type in state 1 exceeds the pivotal type in state 0 even if all types that value market r select market r.

Our main theorem is stated below:

Theorem 3.1. *If the object-to-bidder ratio in market s exceeds $\bar{\kappa}$, then there is no equilibrium sequence that aggregates information in either market. If the object-*

[13] If $l(\theta') \leq 1$, then $\theta^*(\theta') = \theta'$ and the function is decreasing. Otherwise, $F([\theta^*(\theta'), \theta']|0) = F([\theta^*(\theta'), \theta']|1)$ and the implicit function theorem implies that $d\theta^*/d\theta' = f(\theta'|0)\,(l(\theta') - 1)\,/\,f(\theta^*|0)\,(l(\theta^*) - 1)$. The fact that $F([\theta^*(\theta'), \theta']|0) = F([\theta^*(\theta'), \theta']|1)$ and MLRP together imply that $l(\theta') < 1$. Moreover, if $\theta^*(\theta') < \theta'$, then MLRP implies that $l(\theta^*) > 1$. Therefore, $d\theta^*/d\theta' < 0$.

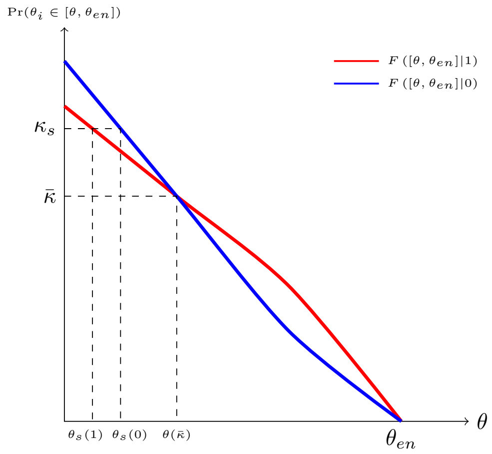

Figure 3.1: The function $F([\theta, \theta_{en}]|v)$ depicts the fraction of types above θ that bid in auction s in state v given that all types $\theta > \theta_{en}$ take the outside option. The cutoff $\bar{\kappa}$ is defined as the value of $F([\theta, \theta_{en}]|v)$ at the point $\theta < \theta_{en}$ where $F([\theta, \theta_{en}]|1)$ and $F([\theta, \theta_{en}]|0)$ cross. If $\kappa_s > \bar{\kappa}$, then the pivotal type in state 0 exceeds the pivotal type in state 1.

to-bidder ratio in market s is less than $\bar{\kappa}$, then information is aggregated in market s along any equilibrium sequence.

The argument for our main theorem shows that information cannot be aggregated along any equilibrium sequence in market s if the order of the pivotal types in this market is reversed whenever all the types that value market r select market r, i.e., if $\kappa_s > \bar{\kappa}$. In other words, self-selection is detrimental to information aggregation when scarcity is sufficiently low, or equivalently, when the object-to-bidder ratio in market s is above the threshold $\bar{\kappa}$. Conversely, information is aggregated in market s along any equilibrium sequence if the order of the pivotal types is preserved even when all the types that value market r select market r, i.e., if scarcity is sufficiently high ($\kappa_s < \bar{\kappa}$).

Before providing some intuition for Theorem 3.1, we describe an intermediate result (Lemma A.8 in the Appendix) that we utilize: if information is aggregated in market s, then the price in market s converges to zero in state $V = 0$ and one in state $V = 1$, i.e., price converges to value. In order to provide an argument for this intermediate result, we first note that Lemma 2.2 implies that there is a bid $b^* > 0$ that separates the support of the limit-price distribution in state 0 from the support of the limit-price distribution in state 1 if information is aggregated in market s.

The first step of the argument that establishes the intermediate result stated above shows that the limit-price distribution's support lies below b^* in state 0 and above b^* in state $V = 1$: Suppose, on the way to a contradiction, that the limit-price distribution's support lies above b^* in state $V = 0$ and below b^* in state $V = 1$. Then, any bidder can ensure that she wins an object only in state $V = 1$ with probability one by submitting a bid equal to b^*. Therefore, any bidder that submits a bid greater than b^* can improve her payoff by instead submitting a bid equal to b^*. So, the limit-price distribution's support cannot lie above b^* in state 0. The second step argues that bids less than b^* must all converge to zero, and therefore the price in state 0 must converge to zero: Any bid less than b^* never wins in state $V = 1$ and therefore any such bid, and in particular, the bid of the pivotal type in state $V = 0$ must converge to zero. The final step concludes that the price in state $V = 1$ must converge to one. If the expected price in state 1 is strictly less than one, then the pivotal type in state 0 could improve her payoff by bidding one instead of following her equilibrium strategy. If she follows her equilibrium strategy, she never wins an object in state $V = 1$ and receives a payoff equal to zero, while under the deviation she wins an object at a price equal to zero in state $V = 0$ and at a price which is strictly less than one in state $V = 1$ with positive probability.

Intuition for why information is not aggregated in market s if $\kappa_s > \bar{\kappa}$: On the way to a contradiction, assume that price converges to value in market s and therefore the payoff of any type that bids in market s is equal to zero. If this is so, then all types that exceed θ_{en} would opt for market r. To see this, observe that if any type $\theta > \theta_{en}$ did not choose market r, then less optimistic types would not choose market r either. Moreover, at the limit, types that exceed θ_{en} face a choice between market s, where their payoff is equal to zero, and market r, where their payoff is positive (in fact, their payoff is equal to $-c$ if $V = 0$ and $1 - c$ if $V = 1$). However, if all types that exceed θ_{en} opt for market r and if $\kappa_s > \bar{\kappa}$, then we find $\theta_s(0) > \theta_s(1)$ (see figure 3.1). If information is aggregated in market s, then $\lim_{n\to\infty} b^n(\theta_s^n(1)) = 1$ and $\lim_{n\to\infty} b^n(\theta_s^n(0)) = 0$ because price converges to value. However, this leads to a contradiction that proves the result. The findings that $\lim_{n\to\infty} b^n(\theta_s^n(1)) = 1$, $\lim_{n\to\infty} b^n(\theta_s^n(0)) = 0$, and $\theta_s(0) > \theta_s(1)$ together contradict that the bidding function is nondecreasing in θ for all n. Intuitively, more pessimistic types opt for market s and there are more of such types in state $V = 0$. Therefore, the auction clears at the bid of a more pessimistic type in state $V = 1$ than in state $V = 0$ and this is incompatible with price converging to value.

Intuition for why information is not aggregated in market r: In market r information aggregation fails for any κ_s in contrast to market s. A similar argument to the one given for market s implies that the price in market r converges to one in state $V = 1$ if information is aggregated. However, if price in market r converges to one in state $V = 1$, then the payoff from bidding in market r is negative for all types and therefore no type would choose this market. But if no type chooses this market, then the price is equal to c in both states and information is not aggregated in market r.

Recall that information is aggregated in an auction if and only if the pivotal types are distinct and they submit distinct bids (no pooling by pivotal types) by Lemma 2.2. Therefore, if $\kappa_s > \bar{\kappa}$, then information aggregation must fail in market s either because the pivotal types are arbitrarily close or because the pivotal types bid in an atom. We construct examples of equilibria where the pivotal types are arbitrarily close and where the pivotal types bid in an atom in section 4 and the appendix, respectively.

Intuition for why information is aggregated in market s if $\kappa_s < \bar{\kappa}$: The definition of $\bar{\kappa}$ implies that $\theta_s(1) > \theta_s(0)$ whenever $\kappa_s < \bar{\kappa}$, i.e., the pivotal types are distinct. Below, we argue that there can be no pooling by pivotal types either whenever $\theta_s(1) > \theta_s(0)$. But then Lemma 2.2 implies that information is aggregated.

To sustain a pool, the highest type that submits the pooling bid (denoted by θ_p) must prefer the pooling bid to a slightly higher bid that wins an object with probability one whenever the price is equal to the pooling bid. Also, the lowest type that submits the pooling bid (denoted by $\underline{\theta}_p$) must prefer the pooling bid to a slightly lower bid that avoids winning an object whenever the price is equal to the pooling bid. In other words, pooling must be incentive compatible for type θ_p and individually rational for type $\underline{\theta}_p$. In the terminology of Lauermann and Wolinsky (2017) (or Pesendorfer and Swinkels (1997)), we say that there is **winner's blessing** at pooling if the probability of winning at the pooling bid is higher when $V = 1$ than when $V = 0$, in other words, if a bidder wins more frequently at pooling when the object's value is high. Similarly, there is **loser's blessing** at pooling if a bidder loses more frequently at pooling when the object's value is low. Put another way, if there is loser's and winner's blessing at pooling, then losing is a signal in favor of $V = 0$ and winning a signal in favor of $V = 1$. The strengths of these two signals determine whether a pooling bid is incentive compatible and individually rational. In particular, the loser's blessing's strength determines the lowest pooling bid that is incentive compatible for type θ_p while the winner's blessing's strength determines the highest pooling bid that is individually rational for type $\underline{\theta}_p$. Our key result that establishes that pooling by pivotal types is not possible shows that if $\theta_s(1) > \theta_s(0)$, then there are bounds on the strength of the loser's and winner's blessing at the pooling bid. These bounds preclude a pooling bid that is both individually rational for type $\underline{\theta}_p$ and incentive compatible for type θ_p thus establishing that pooling by pivotal types is incompatible with equilibrium.

Theorem 3.1 showed that the level of scarcity in the frictionless market (κ_s), together with the cutoff object-to-bidder ratio $\bar{\kappa}$, determines whether information is aggregated. The following remark, which presents comparative statics for $\bar{\kappa}$, further suggests that a frictional market r with little scarcity (i.e., a large object-to-bidder ratio κ_r) is more likely to disrupt information aggregation. Taken together, our analysis identifies the scarcity parameters κ_s and κ_r as key determinants of information aggregation.

Remark 3.1. The ratio $\bar{\kappa}$ is non-increasing in the object-to-bidder ratio in the market r and non-decreasing in the level of frictions c. This is because the type θ_{en} is non-decreasing in c and non-increasing in κ_r. Consequently, $\theta^*(\theta_{en})$ is non-increasing in c and non-decreasing in κ_r. If no type finds it profitable to purchase an object at a price equal to c, i.e., if $c > \Pr(V = 1|\theta)$ for all θ, then $\theta^*(\theta_{en}) = 1$ and $\bar{\kappa} = 1$. If all types are perfectly informed or if $c = 0$, then information is aggregated in both markets whenever $\kappa_s + \kappa_r < 1$ and information is not aggregated in either market if $\kappa_s + \kappa_r > 1$ in both of these cases.[14]

4. Equilibria in the Illustrative Example

In this section, we use the example discussed in the introduction to illustrate how type dependent market selection leads to non-revealing prices in equilibrium. We assume that $\kappa_s + \kappa_r < 1$ and that signals are drawn according to the density function

$$f(\theta|V) = \begin{cases} 3\,(1-g)\,(1-V) & \text{for } \theta \in \mathcal{E}\,(0) := [0, 1/3) \\ 3g & \text{for } \theta \in \mathcal{E}\,(1/2) := [1/3, 2/3] \\ 3\,(1-g)\,V & \text{for } \theta \in \mathcal{E}\,(1) := (2/3, 1] \end{cases}$$

where $g \in [0, 1]$ is the fraction of uninformed types $\theta \in \mathcal{E}\,(1/2)$. Note that all types, other than those in $\mathcal{E}\,(1/2)$, are perfectly informed. Under these assumptions, information aggregation fails in both markets if the object-to-bidder ratio in the frictional market exceeds the fraction of informed bidders (i.e., $\kappa_r > 1 - g$) and information is aggregated in both markets if $\kappa_r < 1 - g$. We will describe

[14] If all types are perfectly informed, then a straightforward computation yields $\bar{\kappa} = 1 - \kappa_r$.

equilibrium sequences for each of these two cases here. Proposition A.1, in the appendix, constructs these sequences and shows that the properties highlighted here hold across all equilibrium sequences.

This example focuses attention on market selection since bidding is relatively simple: Each $\theta \in \mathcal{E}(0)$ bids zero, each $\theta \in \mathcal{E}(1)$ bids one, and each $\theta \in \mathcal{E}(1/2)$ bids $\mathbb{E}[V|Y_m^{n-1}(k_m) = \theta]$, in the market of their choice. Moreover, types in $\mathcal{E}(0)$ never bid in market r because $c > 0$. Therefore, pinning down market selection strategies for the uninformed types and those in $\mathcal{E}(1)$ is sufficient to construct an equilibrium. In the equilibrium sequences that we describe, the mass of types that select market s exceeds κ_s in both states and the mass that separates the two pivotal types (i.e., $\theta_s^n(1)$ and $\theta_s^n(0)$) is equal to the mass of types in $\mathcal{E}(1)$ that bid in market s, i.e., $F_s^n(\mathcal{E}(1)|1)$.

This example's structure also allows us to compute the limit price distribution in market s in closed form using the central limit theorem: the bid of a type, which is z standard deviations from the pivotal type in state $V = 1$, converges to $b_s(z) = \frac{\phi(z)}{\phi(z+x)}/\left(1 + \frac{\phi(z)}{\phi(z+x)}\right)$, where ϕ is the standard normal density, $x := \lim F_s^n(\mathcal{E}(1)|1)/\sigma^n$ is the mass that separates the two pivotal types expressed in standard deviations, and $\sigma^n \approx \sqrt{(1-\kappa_s)\kappa_s/n}$ is the standard deviation. The limit price is less than or equal to $b_s(z)$ with probability $\Phi(z)$ and $\Phi(z+x)$ in states $V = 1$ and $V = 0$, respectively. As x approaches infinity, the prices in state $V = 1$ and $V = 0$ converge to point masses at $p = 1$ and $p = 0$, respectively. As x approaches zero, the price converges to a point mass at $1/2$ in both states.

We now assume $1 - g < \kappa_r$ and describe equilibrium sequences along which information aggregation fails by considering two separate cases: $c > 1/2$ and $c < 1/2$. In both of these cases, the mass of types in $\mathcal{E}(1)$ that chooses market s is positive for each n but converges to zero at the order of $1/\sqrt{n}$. Information aggregation fails in market r due to *insufficient competition:* at the limit the number of bidders in this market is less than the number of objects and the price is equal to the reserve price c in both states with positive probability. Information aggregation fails in market s due to *pivotal types that are arbitrarily close* even though there is sufficient competition. This is because the mass that separates the two pivotal types is chosen at the order of $1/\sqrt{n}$ by construction. The limit price distribution in market s is atomless and strictly increasing over $[0, 1]$ in both states. See figure 4.1 for a depiction of the limit price distributions as a function of x.

Suppose that $c > 1/2$. Under this assumption, no uninformed type chooses

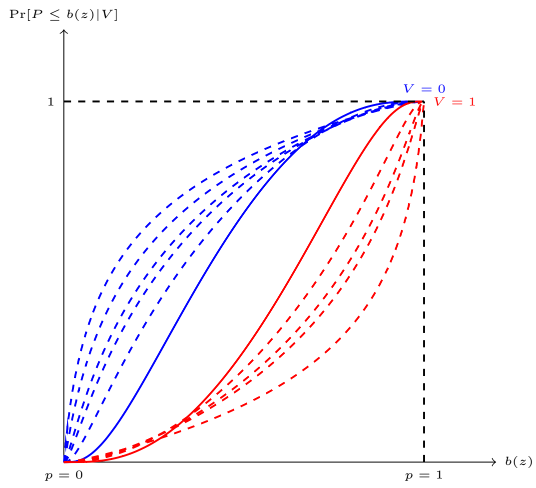

Figure 4.1: Limit cumulative price distributions in market s in states $V = 0$ and $V = 1$. The price distributions' properties imply that $\lim \mathbb{E}[P_s^n|V = 1] = 1 - \lim \mathbb{E}[P_s^n|V = 0]$. If $c > 1/2$, then x is such that $\lim \mathbb{E}[P_s^n|V = 1] = c$. Similarly, if $c < 1/2$, then x is such that $\lim \mathbb{E}[P_s^n|V = 0] = c$. The solid curves depict the cumulative price distributions for $c = 0.6$ or $c = 0.4$, in which case x is approximately equal to one. The dotted curves depict the price distributions as c ranges from 0.6 to 0.8.

market r. Since only types in $\mathcal{E}(1)$ bid in market r, the mass of types that bid in market r is less than κ_r and the price in market r converges to the reserve price c in both states, i.e., the price is uninformative. The construction's main step picks the sequence $\{F_s^n(\mathcal{E}(1)|1)\}$ to ensure that types in $\mathcal{E}(1)$ are indifferent between the two markets for each n. This choice implies that the expect price in market s is also equal to c in state $V = 1$ at the limit, i.e., expected prices are equalized across markets in state $V = 1$.

If $c < 1/2$, then in contrast to the previous case, uninformed types also bid in both markets. The mass of uninformed types that select market r is chosen to ensure that the mass of types in market r converges to κ_r and to a value strictly less than κ_r in states $V = 1$ and $V = 0$, respectively. The price converges to c in state $V = 0$ and converges to a binary random variable that is equal to one and c with probabilities $(1 - 2c)/(1 - c)$ and $c/(1 - c)$, respectively, in state $V = 1$. Hence, the expected price in state $V = 1$ converges to $1 - c$. The construction's main step again chooses the sequence $\{F_s^n(\mathcal{E}(1/2)|1), F_s^n(\mathcal{E}(1)|1)\}$ to ensure that types in $\mathcal{E}(1/2)$ and $\mathcal{E}(1)$ are indifferent between the two markets. These choices imply that the expected limit price in both markets is equal to c and $1-c$ in states $V = 0$ and $V = 1$, respectively, i.e., expected prices in both states are equalized across markets.

Remark 4.1. In this example, information aggregation fails either due to insufficient competition (market r) or because the same set of types determine the price at the limit (market s). More generally, information aggregation in market s can also fail due to pooling by pivotal types and we present an example in the online appendix.

We end this section by assuming $1 - g > \kappa_r$ and describing an equilibrium sequence along which information is aggregated. Along the equilibrium sequence, only types in $\mathcal{E}(1)$ bid in market r and the mass of such types that bid in market r exceeds κ_r in state $V = 1$. Therefore, the price in market r converges to c and 1 in states $V = 0$ and $V = 1$, respectively. A positive mass of types in $\mathcal{E}(1)$ also bid in market s and $x = \lim F_s^n(\mathcal{E}(1)|1)/\sigma^n \to \infty$ along the subsequence that we construct. Therefore, the price in market s converges to 0 and 1 in states $V = 0$ and $V = 1$, respectively.

5. Discussion and Conclusion

The results that we presented in the paper argued that the price in a large, uniform-price, common-value auction may not aggregate all available information if bidders have access to an alternative market that delivers state dependent payoffs. However, we studied only one such instance. There are many other institutional configurations that could result in similar outcomes. For example, market r could instead be (1) A pay-as-you-bid (discriminatory price) auction as in Jackson and Kremer (2007), where all bidders that win an object from the auction pay their own bid, (2) An all-pay-auction as in Chi et al. (2019), or (3) A uniform-price auction where each bidder must pay a positive cost in order to submit a bid as in Murto and Valimaki (2014). The payoff distributions in these alternative specifications have similar properties to the payoff distribution in market r as described by Theorem 3.1: payoffs are negative in state $V = 0$ and positive in state $V = 1$. Our analysis suggests that information aggregation could be hindered also by such market mechanisms.

A. Appendix

Throughout the Appendix, given a sequence of strategies $\mathbf{H} = \{H^n\}_{n=1}^{\infty}$ for a sequence of auctions $\{\Gamma^n\}_{n=1}^{\infty}$, the notation $\Pr^n$ represents the joint probability distribution over states of the world, signal and bid distributions, allocations, market choices, and prices, where this distribution is induced by the strategy H^n. Given a strategy H^n, we denote the payoff to type θ from bidding b in auction s by $u^n(s, b|\theta)$ and this type's payoff by $u^n(\theta)$.

A.1. Bidding Equilibria Suppose participation in market s is exogenously determined by a function $F_s(\cdot)$ that is absolutely continuous with respect to $F(\cdot)$ and $\hat{\Gamma}(F_s)$ is the auction where each type θ is allowed to bid in the auction with probability $a(\theta)$ and is assigned a payoff equal to zero with probability $1 - a(\theta)$. A strategy H is a bidding equilibrium if it is a symmetric Nash equilibrium of the auction $\hat{\Gamma}(F_s)$.

Denote by $\mathcal{E}(\theta') = \{\theta : l(\theta_i = \theta) = l(\theta_i = \theta')\}$ an equivalence class of types that receive signals that generate the same posterior. If $\mathcal{E}(\theta')$ is not a singleton, then H may involve a range of bids given a signal in $\mathcal{E}(\theta')$. However, for any such H there is another strategy, which is pure and increasing on each $\mathcal{E}(\theta')$, such that this strategy yields the same payoff to the player, and is indistinguishable to any other player. Strategies which differ only in their representation over sets $\mathcal{E}(\theta')$ generate the same joint distribution over values, bids, and equilibrium prices. We choose a representation of H which is pure and nondecreasing over equivalence classes $\mathcal{E}(\theta')$.

The following lemma shows that the bids of the pivotal types determine the auction-clearing price of a sufficiently large auction.

Lemma A.1. *Suppose* $\lim \bar{F}_s^n(0|v) > \kappa_s$ *and let* $\underline{\theta}^n$ *denote the type such that* $F_s^n([\underline{\theta}^n, \theta_s^n(v)]|v) = \epsilon$ *and* $\underline{\theta}^n = 0$ *if no such type exist. Similarly, let* $\bar{\theta}^n$ *denote the type such that* $F_s^n([\theta_s^n(v), \bar{\theta}^n]|v) = \epsilon$ *whenever such a type exists. For every* $\epsilon > 0$, $\lim \Pr(P^n \in [b^n(\underline{\theta}^n), b^n(\bar{\theta}^n)]|V = v) = 1$ *where* $b^n(0) = 0$. *Conversely, if* $\lim \bar{F}_s^n(0|v) < \kappa_s$, *then* $\lim \Pr(P^n = 0|V = v) = 1$.

Proof. The LLN implies that $\lim \Pr(Y_s^n(k_s + 1) \geq \underline{\theta}^n|V = v) = 1$ for every $\epsilon > 0$. However, if $Y_s^n(k_s + 1) \geq \underline{\theta}^n$, then $P^n = b^n(Y_s^n(k_s + 1)) \geq b^n(\underline{\theta}^n)$ because b^n is nondecreasing by Lemma 2.1. Therefore, $\Pr(P^n \geq b^n(\underline{\theta}^n)|V = v) \geq \Pr(Y_s^n(k_s + 1) \geq \underline{\theta}^n|V = v)$, and taking limits proves the first part of the claim. We establish $\lim \Pr(Y_s^n(k_s + 1) \leq \bar{\theta}^n|V = v) = 1$ using the same idea. If $\lim \bar{F}_s^n(0|v) < \kappa_s$, then $\lim \Pr(P^n = 0|V = v) = 1$ also follows directly from the LLN. □

A.1.1. *Pooling Calculations.* In this subsection we determine when pooling by pivotal types is incompatible with equilibrium. Given a strategy H, denote by $\Pr(b\ win|P^n = b, V = v, \theta)$ the conditional probability that bidder i wins an object with a bid equal to b given that the auction price is equal to b, the state is equal to v, and bidder i receives a signal equal to θ. Our assumptions that the signals are conditionally independent given V and that H is symmetric together imply that $\Pr(b\ win|P^n = b, V = v, \theta) = \Pr(b\ win|P^n = b, V = v)$. This is because once one conditions on the state, the individual signal of bidder i does not provide any additional information (conditional independence). Moreover, this probability is independent of the identity of the bidder that we consider because we focus on symmetric strategies.

Given a pooling bid b_p^n, let $\theta_p^n = \sup\{\theta : b^n(\theta) = b_p^n\}$, $\underline{\theta}_p^n = \inf\{\theta : b^n(\theta) = b_p^n\}$, and let $\lim \theta_p^n = \theta_p$ and $\lim \underline{\theta}_p^n = \underline{\theta}_p$ whenever these limits exist. The following lemma calculates $\Pr(b_p^n \; wins | P^n = b_p^n, V = v)$ for various cases, and the proof, which involves lengthy computations, is in the online appendix.

Lemma A.2. *If* $\lim \Pr(P^n = b_p^n | v) > 0$*, then there is a constant* $C > 0$ *such that*

$$\Pr(b_p^n(\theta^n) \; lose | P^n = b_p^n, V = v) \geq C \frac{\max\{\kappa_s - F_s^n(\underline{\theta}_p^n|1), 1/\sqrt{n}\}}{F_s^n([\underline{\theta}_p^n, \theta_p^n]|1)}$$

for all sufficiently large n*. If* $\lim F_s^n([\underline{\theta}_p^n, \theta_p^n]|v) > 0$*, then*

$$\lim \Pr(b_p^n \; win | P^n = b_p^n, V = v) = \lim \frac{\kappa_s - \bar{F}_s^n(\theta_p^n|v)}{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)}.$$

If $\lim \Pr(P^n \geq b_p^n | v = 0)$*, then*

$$\lim \Pr(b_p^n \; lose | P^n = b_p^n, V = v) / \frac{\overline{F}_s^n(\underline{\theta}_p^n|v)(1 - \bar{F}_s(\underline{\theta}_p^n|v))}{n F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)(\kappa_s - \bar{F}_s^n(\underline{\theta}_p^n|v))} = 1.$$

Lemma A.3. *Fix a sequence of bidding equilibria* $\mathbf{H}$*. If* $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$ *or if* $F_s(1|1) \geq F_s(1|0)$ *and* $F_s(1|1) > \kappa_s$*, then there is no pooling by pivotal types.*

Proof. We will argue that if $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$, then pooling by pivotal types is incompatible with equilibrium. At the end of the proof we show that $F_s(1|1) \geq F_s(1|0)$ and $F_s(1|1) > \kappa_s$ imply $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$.

The fact that $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$ implies $\theta_s(1) > \theta_s(0)$ and $F_s(\theta_s(1)|0) > F_s(\theta_s(0)|0)$. Pooling by pivotal types implies that $F_s(\underline{\theta}_p|v) \leq F_s(\theta_s(0)|v) < F_s(\theta_s(1)|v) \leq F_s(\theta_p|v)$. We will show that pooling by pivotal types is incompatible with equilibrium behavior in the following three cases: (1) $F_s(\underline{\theta}_p|v) < F_s(\theta_s(0)|v)$ and $F_s(\theta_p|v) > F_s(\theta_s(1)|v)$; (2) $F_s(\theta_p|v) = F_s(\theta_s(1)|v)$ and $F_s(\underline{\theta}_p|v) < F_s(\theta_s(0)|v)$; and (3) $F_s(\underline{\theta}_p|v) = F_s(\theta_s(0)|v)$.

Case 1: $F_s(\underline{\theta}_p|v) < F_s(\theta_s(0)|v)$ and $F_s(\theta_p|v) > F_s(\theta_s(1)|v)$. For type θ_p bidding b_p instead of bidding slightly above the pooling bid is incentive-compatibility: $(1 - b_p) l(\theta_p) \lim \Pr(P^n = b_p^n, b_p^n \quad win | V = 1) - b_p \lim \Pr(P^n = b_p^n, b_p^n \quad win | V =$

$0) \geq (1-b_p)l(\theta_p) - b_p$.Therefore,

$$\frac{b_p}{1-b_p} \geq l(\theta_p)\frac{\lim \Pr(P^n = b_p^n, b_p^n \quad loses|V=1)}{\lim \Pr(P^n = b_p^n, b_p^n \quad loses|V=0)}.$$

Pooling is individually rational for type $\underline{\theta}_p$: $(1-b_p)l(\underline{\theta}_p)\lim \Pr(P^n = b_p^n, b_p^n \quad win|V = 1) - b_p \lim \Pr(P^n = b_p^n, b_p^n \quad win|V=0) \geq 0$. Therefore,

$$\frac{b_p}{1-b_p} \leq l(\underline{\theta}_p)\frac{\lim \Pr(P^n = b_p^n, b_p^n \quad wins|V=1)}{\lim \Pr(P^n = b_p^n, b_p^n \quad wins|V=0)}.$$

Combining the incentive compatibility and individual rationality constraints and substituting in using by Lemma A.2, we obtain

$$l(\underline{\theta}_p)\frac{F_s(\theta_p|1) - F_s(\theta_s(1)|1)}{F_s(\theta_p|0) - F_s(\theta_s(0)|0)} \geq l(\theta_p)\frac{F_s(\theta_s(1)|1) - F_s(\underline{\theta}_p|1)}{F_s(\theta_s(0)|0) - F_s\left(\underline{\theta}_p|0\right)},$$

which is not possible because $l(\underline{\theta}_p) \leq \frac{F_s(\theta_s(0)|1) - F_s(\underline{\theta}_p|1)}{F_s(\theta_s(0)|0) - F_s(\underline{\theta}_p|0)} < \frac{F_s(\theta_s(1)|1) - F_s(\underline{\theta}_p|1)}{F_s(\theta_s(0)|0) - F_s(\underline{\theta}_p|0)}$ and because $\frac{F_s(\theta_p|1) - F_s(\theta_s(1)|1)}{F_s(\theta_p|0) - F_s(\theta_s(0)|0)} < \frac{F_s(\theta_p|1) - F_s(\theta_s(1)|1)}{F_s(\theta_p|0) - F_s(\theta_s(1)|0)} \leq l(\theta_p)$ by MLRP.

Case 2: If $F_s(\theta_p|v) = F_s(\theta_s(1)|v)$ and $F_s(\underline{\theta}_p|v) < F_s(\theta_s(0)|v)$, then Lemma A.2 implies that $\lim \Pr(P^n = b_p^n, b_p^n \; wins|V=1) = 0$ and $\lim \Pr(P^n = b_p^n, b_p^n \; wins|V = 0) > 0$. However, then pooling cannot be sustained by Lemma 7 and Corollary 3 in Pesendorfer and Swinkels (1997).

Case 3: If $F_s(\underline{\theta}_p|v) = F_s(\theta_s(0)|v)$, then Lemma A.2 implies that $\lim \Pr(P^n = b_p^n, b_p^n \; wins|V=0) = 1$ and $\lim \Pr(P^n = b_p^n, b_p^n \; wins|V=1) < 1$ again showing that pooling cannot be sustained by Lemma 7 and Corollary 3 in Pesendorfer and Swinkels (1997).

We conclude the proof by arguing that $F_s(1|1) \geq F_s(1|0)$ and $F_s(1|1) > \kappa_s$ together imply that $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$. On the way to a contradiction assume $F_s(\theta_s(1)|1) \leq F_s(\theta_s(0)|1)$. Note $F_s(1|1) > \kappa_s$ implies $0 < F_s(\theta_s(1)|1) \leq F_s(\theta_s(0)|1)$. Our assumption $F_s(\theta_s(1)|1) \leq F_s(\theta_s(0)|1)$ and MLRP together imply that $1 \geq \bar{F}_s(\theta_s(1)|1)/\bar{F}_s(\theta_s(1)|0) > F_s(\theta_s(1)|1)/F_s(\theta_s(1)|0)$. However, $F_s(1|v) = \bar{F}_s(\theta_s(1)|v) + F_s(\theta_s(1)|v)$, $\bar{F}_s(\theta_s(1)|1) \leq \bar{F}_s(\theta_s(1)|0)$, and $F_s(\theta_s(1)|1) < F_s(\theta_s(1)|0)$ together imply that $F_s(1|1) < F_s(1|0)$ leading to a contradiction. □

The following lemma shows that there cannot be a pooling bid that occurs with positive probability in state $V = 1$ and probability zero in state $V = 0$ if the pivotal types are distinct.

Lemma A.4. *Fix a sequence of bidding equilibria* $\mathbf{H}$ *and assume* $\sqrt{n}(F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)) \to \infty$. *There is no sequence of pooling bids* b_p^n *such that* $\lim \Pr(P^n = b_p^n|V = 1) > 0$ *and* $\lim \Pr(P^n \geq b_p^n|V = 0) = 0$.

Proof. We will show that $\lim \frac{\Pr(b^n\ lose|P^n=b_p^n,V=0)}{\Pr(b^n\ lose|P^n=b_p^n,V=1)} = 0$ which implies that pooling cannot be sustained for sufficiently large n by Lemma 7 and Corollary 3 in Pesendorfer and Swinkels (1997). Lemma A.2 gives that

$$\lim \frac{\Pr(b^n\ lose|P^n = b_p^n, 0)}{\Pr(b^n\ lose|P^n = b_p^n, 1)} \leq$$
$$\lim \frac{F_s^n([\underline{\theta}_p^n, \theta_p^n]|1)}{F_s^n([\underline{\theta}_p^n, \theta_p^n]|0)} \frac{C\overline{F}_s^n(\underline{\theta}_p^n|0)(1 - \bar{F}_s(\underline{\theta}_p^n|0))}{n(\kappa_s - \bar{F}_s^n(\underline{\theta}_p^n|0)) \max\{\kappa_s - F_s^n(\underline{\theta}_p^n|1), 1/\sqrt{n}\}}$$

where $C \in (0, \infty)$. However, $F_s^n([\underline{\theta}_p^n, \theta_p^n]|1)/F_s^n([\underline{\theta}_p^n, \theta_p^n]|0) \leq 1/\eta$ by Lemma A.5, $n \max\{\kappa_s - F_s^n(\underline{\theta}_p^n|1), 1/\sqrt{n}\} \geq \sqrt{n}$, $\overline{F}_s^n(\underline{\theta}_p^n|0)(1 - \bar{F}_s(\underline{\theta}_p^n|0)) \leq 1$, and $\lim \sqrt{n}(\kappa_s - \bar{F}_s^n(\underline{\theta}_p^n|0)) = \infty$ (because $\lim \Pr(P^n \geq b_p^n|V = 0) = 0$). Therefore,

$$\lim \Pr(b^n\ lose|P^n = b_p^n, 0)/\Pr(b^n\ lose|P^n = b_p^n, 1) \leq$$
$$\lim 1/(C\eta\sqrt{n}(\kappa_s - \bar{F}_s^n(\underline{\theta}_p^n|0))) = 0.$$

□

A.1.2. *Information content of being pivotal.* In this subsection, we provide bounds for the ratio $l(Y^n(k_s + 1) = \theta^n) = \Pr(Y^n(k_s + 1) = \theta^n|V = 1)/\Pr(Y^n(k_s + 1) = \theta^n|V = 0)$, i.e., the information content of the event of being pivotal. The results we present below show that the event of being pivotal provides only bounded amounts of information for the types that set the price if the pivotal types are arbitrarily close.

We begin with the following lemma that outlines the implication of our assumption that there are no arbitrarily informative signals.

Lemma A.5. *For any interval* $I \subset [0, 1]$,

$$F_s^n(I|V = 1) \in [\eta F_s^n(I|V = 0), \frac{F_s^n(I|V = 0)}{\eta}].$$

Thus, $\sqrt{n}(F_s^n(\theta_s^n(1)|0) - F_s^n(\theta_s^n(0)|0)) < \infty$ *iff* $\sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| < \infty$.

Proof. To see this, note $F_s^n(I|1) = \int_I a(\theta)f(\theta|1)d\theta = \int_I a(\theta)f(\theta|0)l(\theta)d\theta$ and

$\eta F_s^n(I|0) = \eta \int_I a(\theta) f(\theta|0) d\theta \leq \int_I a(\theta) f(\theta|0) l(\theta) d\theta \leq \frac{1}{\eta} \int_I a(\theta) f(\theta|0) d\theta = \frac{1}{\eta} F_s^n(I|0)$ because $l(\theta) \in (\eta, 1/\eta)$ for $\theta \in [0,1]$. □

The probability that a particular type θ is pivotal (i.e., $Y_s^n(k_s+1) = \theta$) can be approximated using the central limit theorem. If $\lim \frac{n\kappa_s - n\bar{F}_s^n(\theta|v)}{\sqrt{n\kappa_s(1-\kappa_s)}} = a$, then $Bi(k_s; n, \bar{F}_s^n(\theta|v)) \to \Phi(a)$ where Bi and Φ denote the binomial and standard normal cumulative distributions, respectively. Moreover, if we let $p = \bar{F}_s^n(\theta|v)$, then

$$bi(k_s; n, p) = \binom{n}{k_s} p^{k_s}(1-p)^{n-k_s} = \frac{1+\delta_n(p)}{\sqrt{2\pi n\kappa_s(1-\kappa_s)}} \phi\left(\frac{k_s - np}{\sqrt{\kappa_s(1-\kappa_s)n}}\right) \quad \text{(A.1)}$$

where bi and ϕ denote the binomial and standard normal densities, respectively; and $\lim_{n\to\infty} \sup_{p:|np-k_s|<n^t} \delta_n(p) = 0$ for $t < 2/3$ (see Lesigne (2005, Proposition 8.2)). In the following two lemmata, we use these convergence results and show that if the price is set by a type θ that is within finitely many standard deviations of both pivotal types, then the information that this type gets from being pivotal is bounded.

For any $\theta \in [0,1]$ and $v = 0, 1$ define

$$z_v^n(\theta) := \frac{k_s - (n-1)\bar{F}_s^n(\theta|v)}{\sqrt{(n-1)\kappa_s(1-\kappa_s)}}.$$

Lemma A.6. *Pick a sequence of types $\{\theta^n\}$ that bid in market s. Assume that $\lim z_v^n(\theta^n) = z_v$ for $v = 0, 1$ and $\lim l(\theta^n) = \rho$. For any $\delta > 0$, there exists an N such that for all $n > N$ we have $\rho(1-\delta)\phi(z_1)/\phi(z_0) \leq l(Y^n(k_s+1) = \theta^n) \leq \rho(1+\delta)\phi(z_1)/\phi(z_0)$. Therefore, $l(Y^n(k_s+1) = \theta^n) \to \phi(z_1)/\phi(z_0)\rho$.*

Proof. A direct computation shows that $l(Y^n(k_s+1) = \theta^n) = l(\theta^n) \frac{bi(k_s; n-1, \bar{F}_s^n(\theta^n|1))}{bi(k_s; n-1, \bar{F}_s^n(\theta^n|0))}$. Eq. (A.1) implies that for any $\delta > 0$, there exists an N such that

$$(1-\delta)\phi(z_1^n(\theta^n))/\phi(z_0^n(\theta^n)) \leq bi(k_s; n-1, \bar{F}_s^n(\theta^n|1))/bi\left(k_s; n-1, \bar{F}_s^n(\theta^n|0)\right) \leq (1+\delta)\phi(z_1^n(\theta^n))/\phi(z_0^n(\theta^n))$$

for all $n > N$. Our assumption that $\lim z_v^n(\theta^n) = z_v$ and $k_s/(n-1) \to \kappa_s$ together establish that $\lim \sqrt{n}|\bar{F}_s^n(\theta^n|v) - \kappa_s| < \infty$ for $v = 0, 1$. The fact that $\phi(z_v^n(\theta))$ is a continuous functions of θ implies that for any $\delta > 0$, there exists an N such that for all $n > N$ we have $\rho(1-\delta)\phi(z_1)/\phi(z_0) \leq l(Y^n(k_s+1) = \theta^n) \leq \rho(1+\delta)\phi(z_1)/\phi(z_0)$. □

Lemma A.7. *Assume* $\lim \sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| = x < \infty$ *and* $\lim \sqrt{n}(\kappa_s - \bar{F}_s^n(0|0)) = -\infty$. *Suppose* θ_y^n *is a the type such that* $F_s^n([\theta_y^n, \theta_s^n(0)]|0) = \sqrt{\kappa_s(1-\kappa_s)/n}$. *For any* $\delta > 0$, *there exists an* N *such that for all* $n > N$ *and for any interval* $[a,b] \subset [\theta_y^n, \theta_s^n(0)]$ *such that* $F_s^n([a,b]|0) > 0$ *we have* $\eta(1-\delta)\phi(x+y/\eta)/\phi(0) \leq l(Y^n(k_s+1) \in [a,b]) \leq (1+\delta)\phi(0)/\eta\phi(y)$

Proof. Suppose, without loss of generality, that $\lim \frac{\sqrt{n}(F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1))}{\sqrt{\kappa_s(1-\kappa_s)}} \geq 0$. Note that if $\lim \sqrt{n}(\kappa_s - \bar{F}_s^n(0|0)) = -\infty$, then $\lim \sqrt{n}F_s(\theta_s^n(0)|0) = \infty$ and the interval $\left[\theta_y^n, \theta_s^n(0)\right]$ is well defined for all sufficiently large n. For any sequence $\{\theta^n\}$ such that $\theta^n \in [\theta_y^n, \theta_s^n(0)]$ for every n, we have $\lim z_v^n(\theta^n) = \lim \sqrt{n}(\kappa_s - \bar{F}_s^n(\theta^n|v))/\sqrt{\kappa_s(1-\kappa_s)}$. Also, $l(\theta) \in [\eta, 1/\eta]$ (no arbitrarily informative signals), $\lim z_1^n(\theta^n) \in [-x-y/\eta, 0]$, and $\lim z_0^n(\theta^n) \in [-y, 0]$. Therefore, Lemma A.6 implies that for any $\delta > 0$, there exists an N such that for all $n > N$ and any $\theta \in [\theta_y^n, \theta_s^n(0)]$

$$(1-\delta)\frac{\phi(x+y/\eta)}{\phi(0)} \leq \frac{bi\left(k_s; n-1, \bar{F}_s^n(\theta|1)\right)}{bi\left(k_s; n-1, \bar{F}_s^n(\theta|0)\right)} \leq (1+\delta)\frac{\phi(0)}{\phi(y)}.$$

Thus using the fact that $l(\theta) \in [\eta, 1/\eta]$, we conclude that

$$\eta(1-\delta)\phi(x+y/\eta)/\phi(0) \leq l(Y^n(k_s+1) \in [a,b]) \leq (1+\delta)\phi(0)/\eta\phi(y)$$

for any interval $[a,b] \subset [\theta_y^n, \theta^n(0)]$. □

A.1.3. *Proof of Information Aggregation Lemma 2.2.*

Proof of Lemma 2.2. First we argue that if $\mathbf{H}$ aggregates information, then there is no pooling by pivotal types and the pivotal types are distinct. Note that if there is pooling by pivotal types, then $\mathbf{H}$ does not aggregate information by definition.[15]

We will argue that if $\mathbf{H}$ aggregates information, then the pivotal types are distinct ($\lim \sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| = \infty$). Suppose the pivotal types are arbitrarily close, i.e., $\liminf_n \sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| < \infty$. Then there exists a subsequence $\{n^k\}$ such that $\lim_{n^k} \sqrt{n}|F_s^{n^k}(\theta_s^{n^k}(1)|1) - F_s^{n^k}(\theta_s^{n^k}(0)|1)| = x < \infty$. We will show that information is not aggregated along this subsequence, which, with a slight abuse of notation, we index by n . Recall that $\bar{F}_s^n(0|0)$ is the fraction of types who bid in market s in state 0. In the next two claims, we will show 1) If $\lim \sqrt{n}(\kappa_s - \bar{F}_s^n(0|0)) > -\infty$, then $\mathbf{H}$ does not aggregate information; and 2) If $\lim \sqrt{n}(\kappa_s - \bar{F}_s^n(0|0)) = -\infty$, then $\mathbf{H}$ does not aggregate information.

[15] This is because $\lim \Pr\left(P^n = b_p^n|V = v\right) > 0$ for $v = 0, 1$, i.e., the auction price is equal to the pooling bid with strictly positive probability in both states.

Therefore, we will conclude that if the pivotal types are arbitrarily close, then $\mathbf{H}$ does not aggregate information establishing our claim.

Claim A.1. If $\lim \sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| < \infty$ and $\lim \sqrt{n}(\kappa_s - \bar{F}_s^n(0|0)) > -\infty$, then $\mathbf{H}$ does not aggregate information.

Proof. Suppose $\lim \sqrt{n}\,|F_s^n\,(\theta_s^n(1)|1) - F_s^n\,(\theta_s^n(0)|1)| < \infty$. We will show that the price is equal to zero with strictly positive probability in both states and therefore $\mathbf{H}$ does not aggregate information. Suppose that $\sqrt{n}(\kappa_s - \bar{F}_s^n\,(0|0))/\sigma \to x > -\infty$ where x is possibly equal to $+\infty$ and where $\sigma := \sqrt{\kappa_s(1-\kappa_s)}$. The central limit theorem implies that the number of goods in the auction exceeds the number of bidders with positive probability if $V = 0$, and $\lim_n \Pr\left(Y_s^n\,(k_s+1) = 0|V=0\right) = \Phi\,(x) > 0$.

Below we argue that $\lim \sqrt{n}\,|F_s^n\,(\theta_s^n(1)|1) - F_s^n\,(\theta_s^n(0)|1)| < \infty$ and $\sqrt{n}(\kappa_s - \bar{F}_s^n(0|0))/\sigma \to x > -\infty$ together imply that $\sqrt{n}(\kappa_s - \bar{F}_s^n\,(0|1))/\sigma \to x' > -\infty$. But if $\sqrt{n}(\kappa_s - \bar{F}_s^n\,(0|1))\sigma \to x' > -\infty$, then applying the central limit theorem once again we find $\lim_n \Pr(Y_s^n\,(k_s+1) = 0|V=1) = \Phi\,(x') > 0$ and therefore $\lim_n \Pr\left(P^n = 0|V=1\right) \geq \Phi\,(x') > 0$. However, $\lim_n \Pr\left(P^n = 0|V=v\right) > 0$ for $v = 0, 1$ and $\lim_n l(P^n = 0) = \Phi\,(x')\,/\Phi\,(x) \in (0,\infty)$ contradicts that $\mathbf{H}$ aggregates information.

We argue that

$$\sqrt{n}(\kappa_s - \bar{F}_s^n\,(0|0))/\sigma = \sqrt{n}\left(\kappa_s - \left(F_s^n(\theta_s^n(0)|0) + \bar{F}_s^n\,(\theta_s^n(0)|0)\right)\right)/\sigma \to x > -\infty$$

implies $\lim \sqrt{n}F_s(\theta_s^n(0)|0) < \infty$ and therefore $\lim \sqrt{n}F_s(\theta_s^n(0)|1) < \infty$. By definition we have $\bar{F}_s^n\,(\theta_s^n(0)|0) = \kappa_s$ if $\kappa_s \leq \bar{F}_s^n\,(0|0)$ and $\theta_s^n(0) = 0$ otherwise. Therefore, $\lim \sqrt{n}(\kappa_s - \bar{F}_s^n\,(0|0)) = -\infty$ if and only if $\lim \sqrt{n}F_s(\theta_s^n(0)|0) = \infty$. Hence, $\sqrt{n}(\kappa_s - \bar{F}_s^n\,(0|0))/\sigma \to x > -\infty$ implies that $\lim \sqrt{n}F_s(\theta_s^n(0)|0) < \infty$ and hence $\lim \sqrt{n}F_s(\theta_s^n(0)|1) < \infty$ by Lemma A.5.

We now show that $\lim \sqrt{n}(\kappa_s - \bar{F}_s^n(0|0)) = x > -\infty$ implies $\lim \sqrt{n}(\kappa_s - \bar{F}_s^n(0|1)) > -\infty$. We argued in the previous paragraph that $\sqrt{n}(\kappa_s - \bar{F}_s^n(0|1))/\sigma \to -\infty$ if and only if $\lim \sqrt{n}F_s(\theta_s^n(1)|1) = \infty$. However, if $\lim \sqrt{n}F_s(\theta_s^n(1)|1) = \infty$, then $\lim \sqrt{n}F_s^n([\theta_s^n(0), \theta_s^n(1)]|1) = \infty$ because $\lim \sqrt{n}F_s(\theta_s^n(0)|0) < \infty$ and because $\lim \sqrt{n}F_s(\theta_s^n(0)|1) < \infty$. But this contradicts $\lim \sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| < \infty$. Hence, $\sqrt{n}(\kappa_s - \bar{F}_s^n(0|1))/\sigma \to x'$ for some $x' > -\infty$ which is possibly equal to $+\infty$. □

We now turn to the case where $\lim \sqrt{n}(\kappa_s - \bar{F}_s^n\,(0|0)) = -\infty$. Pick any $y > 0$

and let θ_y^n denote the type such that

$$F_s^n([\theta_y^n, \theta_s^n(0)]|0) = \sigma y/\sqrt{n} \tag{A.2}$$

when such a type exists. Observe that $\theta_y^n < \theta_{2y/3}^n < \theta_{y/3}^n < \theta^n(0)$ and $F_s^n([\theta_{2y/3}^n, \theta_{y/3}^n]|0) = \sigma y/3/\sqrt{n}$ by the definition of these types given in Eq. (A.2). Let $A^n := \{p : p = b^n(\theta), \theta \in [\theta_{2y/3}^n, \theta_{y/3}^n]\}$. The central limit theorem implies that $\lim \Pr(Y^n(k+1) \in [\theta_{2y/3}^n, \theta_{y/3}^n]|V) = 0 = \Phi(2y/3) - \Phi(y/3) > 0$. Also, $\Pr(P^n \in A^n|V = 0) \geq \Pr(Y^n(k+1) \in [\theta_{2y/3}^n, \theta_{y/3}^n]|V) = 0$ because $P^n = b^n(Y^n(k+1))$. The inequality above does not necessarily hold as an equality because types other than those $[\theta_{2y/3}^n, \theta_{y/3}^n]$ may also choose a bid in A^n. Lemma A.7 implies that $\lim \Pr(Y^n(k+1) \in [\theta_{2y/3}^n, \theta_{y/3}^n]|V = 1) \geq \frac{\phi(x+y/\eta)}{\phi(0)}\eta(\Phi(2y/3) - \Phi(y/3)) > 0$. Therefore $\lim \Pr(P^n \in A^n|V = 1) \geq \frac{\phi(x+y/\eta)}{\phi(0)}\eta(\Phi(2y/3) - \Phi(y/3)) > 0$.

Claim A.2. If $\lim \sqrt{n}|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| = x < \infty$ and $\lim \sqrt{n}(\kappa_s - \bar{F}_s^n(0|0)) = -\infty$, then **H** does not aggregate information.

Proof. We will argue that there exists an $\epsilon > 0$ such that $l(P^n = p) \in (\epsilon, 1/\epsilon)$ for any $p \in A^n$ and any n sufficiently large. However, this together with the facts that $\Pr(P^n \in A^n|V = 1) > 0$ and $\Pr(P^n \in A^n|V = 0) > 0$ imply that **H** does not aggregate information.

Pick any $\delta > 0$. For any $\theta^* \in [\theta_{2y/3}^n, \theta_{y/3}^n]$ that bids in market s with positive probability, we have either 1) $\{\theta : b^n(\theta) = b^n(\theta^*)\} \subset [\theta_y^n, \theta^n(0)]$ or 2) $\{\theta : b^n(\theta) = b^n(\theta^*)\} \not\subseteq [\theta_y^n, \theta^n(0)]$. Moreover, the fact that the bidding function is monotone implies that the set $\{\theta : b^n(\theta) = b^n(\theta^*)\}$ is either a singleton or an interval.

If $\{\theta : b^n(\theta) = b^n(\theta^*)\} \subset [\theta_y^n, \theta^n(0)]$, then Lemma A.7 implies that

$$(1-\delta)\frac{\phi(x+y/\eta)}{\phi(0)}\eta \leq l(Y^n(k+1) \in \{\theta : b^n(\theta) = b^n(\theta^*)\}) \leq (1+\delta)\frac{\phi(0)}{\phi(y)}\frac{1}{\eta}$$

for all $n > N(\delta)$.[16] Therefore,

$$\eta(1-\delta)\phi(x+y/\eta)/\phi(0) \leq l(P^n = b^n(\theta^*)) \leq (1+\delta)\phi(0)/\phi(y)\eta$$

for all $n > N(\delta)$.

If, on the other hand, $\{\theta : b^n(\theta) = b^n(\theta^*)\} \not\subseteq [\theta_y^n, \theta^n(0)]$, then either $[\theta_y^n, \theta_{2y/3}^n] \subset \{\theta : b^n(\theta) = b^n(\theta^*)\}$ or $[\theta_{y/3}^n, \theta^n(0)] \subset \{\theta : b^n(\theta) = b^n(\theta^*)\}$ because the set $\{\theta :$

[16] Observe that $N(\delta)$ is independent of θ^* and the set $[\theta_{2y/3}^n, \theta_{y/3}^n]$.

$b^n(\theta) = b^n(\theta^*)\}$ is an interval that extends beyond $[\theta^n_y, \theta^n(0)]$. Therefore, $\Pr(P^n = b^n(\theta^*)|V = v) \geq \Pr(Y^n(k+1) \in [\theta^n_{y/3}, \theta^n(0)]|V = v)$ or $\Pr(P^n = b^n(\theta^*)|V = v) \geq \Pr(Y^n(k+1) \in [\theta^n_y, \theta^n_{2y/3}]|V = v)$. The central limit theorem implies that $(1-\delta)(\Phi(y/3) - 1/2) \leq \Pr(Y^n(k+1) \in [\theta^n_{y/3}, \theta^n(0)]|V = 0)$ and$(1-\delta)(\Phi(y) - \Phi(2y/3)) \leq \Pr(Y^n(k+1) \in [\theta^n_y, \theta^n_{2y/3}]|V = 0)$ for all for all $n > N(\delta)$. Moreover, Lemma A.7 implies that $(1-\delta)\phi(x+y/\eta)\eta(\Phi(y/3) - 1/2)/\phi(0) \leq \Pr(Y^n(k+1) \in [\theta^n_{y/3}, \theta^n(0)]|V = 1)$ and

$$(1-\delta)\phi(x+y/\eta)\eta(\Phi(y) - \Phi(2y/3))/\phi(0) \leq \Pr(Y^n(k+1) \in [\theta^n_y, \theta^n_{2y/3}]|V = 1)$$

for all $n > N(\delta)$. Therefore, $(1-\delta)\phi(x+y/\eta)C\eta/\phi(0) \leq l(P^n = b^n(\theta^*)) \leq 1/(1-\delta)C$ for all for all $n > N(\delta)$ where $C = \min\{\Phi(y/3) - 1/2, \Phi(y) - \Phi(2y/3)\}$. Hence picking ϵ such that $\epsilon < \phi(x+y/\eta)\eta C/\phi(0)$, $\epsilon < C$ and $1/\epsilon > \phi(0)/\phi(y)\eta$ establishes that **H** does not aggregate information. □

We now argue that if there is no pooling by pivotal types and if the pivotal types are distinct, then information is aggregated along a sequence **H**. Denote by $v \in \{0,1\}$ the state where the pivotal type is largest and by v' the other state. Our assumption that the pivotal types are distinct implies that $\sqrt{n}F^n_s([\theta^n_s(v'), \theta^n_s(v)]|0) \to \infty$. For any $\epsilon \in (0, 1/2)$ define

$$\bar{\theta}^n_\epsilon := \min\{\theta : \Pr(Y^n_s(k+1) \leq \theta|V = v)\} = \epsilon, \tag{A.3}$$

$\underline{\theta}^n_\epsilon := \max\{\theta : \Pr(Y^n_s(k+1) \geq \theta|V = v') = \epsilon\}$, and $b^n_\epsilon := (b^n(\underline{\theta}^n_\epsilon) + b^n(\bar{\theta}^n_\epsilon))/2$. These definitions imply that $\theta^n_s(v') < \underline{\theta}^n_\epsilon < \bar{\theta}^n_\epsilon < \theta^n_s(v)$ for sufficiently large n. This is because $\lim \sqrt{n}F^n_s\left(\left[\bar{\theta}^n_\epsilon, \theta^n_s(v)\right]|V = v\right) \in (0,\infty)$ and $\lim \sqrt{n}F^n_s\left([\theta^n_s(v'), \underline{\theta}^n_\epsilon]|V = v'\right) \in (0,\infty)$ by the LLN and $\sqrt{n}F^n_s([\theta^n_s(v'), \theta^n_s(v)]|0) \to \infty$.

We prove the result through the three claims given below. We first argue that $\Pr(Y^n_s(k+1) \leq \underline{\theta}^n_\epsilon|v) \to 0$ and $\Pr(Y^n_s(k+1) \geq \bar{\theta}^n_\epsilon|v') \to 0$ (Claim A.3). We then show that the types $\underline{\theta}^n_\epsilon$ and $\bar{\theta}^n_\epsilon$ submit distinct bids and therefore $b^n(\underline{\theta}^n_\epsilon) < b^n(\bar{\theta}^n_\epsilon)$ (Claim A.4). We complete the proof by showing that the bid distribution is state v lies above b^n_ϵ and the bid distribution in state v' lies below b^n_ϵ with probability converging to one, i.e., b^n_ϵ separates the two bid distributions (Claim A.5).

Claim A.3. If $\sqrt{n}F^n_s([\theta^n_s(v'), \theta^n_s(v)]|0) \to \infty$, then $\Pr(Y^n_s(k+1) \leq \underline{\theta}^n_\epsilon|v) \to 0$ and $\Pr(Y^n_s(k+1) \geq \bar{\theta}^n_\epsilon|v') \to 0$.

Proof. Note $\sqrt{n}F_s^n([\underline{\theta}_\epsilon^n, \bar{\theta}_\epsilon^n]|0) \to \infty$ because

$$\lim \sqrt{n}F_s^n([\theta_s^n(v'), \theta_s^n(v)]|0) = \lim \sqrt{n}(F_s^n([\theta_s^n(v'), \underline{\theta}_\epsilon^n]|0) \\ + F_s^n([\underline{\theta}_\epsilon^n, \bar{\theta}_\epsilon^n]|0) + F_s^n([\bar{\theta}_\epsilon^n, \theta_s^n(v)]|0)$$

$\lim \sqrt{n}F_s^n\left([\theta_s^n(v'), \theta_s^n(v)]\,|0\right) = \infty$, $\lim \sqrt{n}F_s^n([\bar{\theta}_\epsilon^n, \theta_s^n(v)]|0) \in (0, \infty)$ and

$$\lim \sqrt{n}F_s^n([\theta_s^n(v'), \underline{\theta}_\epsilon^n]|0) \in (0, \infty).$$

Moreover, $\sqrt{n}F_s^n([\underline{\theta}_\epsilon^n, \theta_s^n(v)]|v) \to \infty$ and $\sqrt{n}F_s^n([\theta_s^n(v'), \bar{\theta}_\epsilon^n]|v') \to \infty$ follow immediately from $\sqrt{n}F_s^n([\underline{\theta}_\epsilon^n, \theta_s^n(v)]|v) \geq \sqrt{n}F_s^n([\underline{\theta}_\epsilon^n, \bar{\theta}_\epsilon^n]|v)$ and $\sqrt{n}F_s^n([\theta_s^n(v'), \bar{\theta}_\epsilon^n]|v') \geq \sqrt{n}F_s^n([\underline{\theta}_\epsilon^n, \bar{\theta}_\epsilon^n]|v')$. Finally, the LLN implies that $\Pr(Y_s^n(k+1) \leq \underline{\theta}_\epsilon^n|v) \to 0$ and $\Pr(Y_s^n(k+1) \geq \bar{\theta}_\epsilon^n|v') \to 0$. □

Claim A.4. If the pivotal types are distinct and there is no pooling by pivotal types, then $b^n\left(\underline{\theta}_\epsilon^n\right) < b^n\left(\bar{\theta}_\epsilon^n\right)$ for all sufficiently large n.

Proof. Monotonicity implies $b^n\left(\underline{\theta}_\epsilon^n\right) \leq b\left(\bar{\theta}_\epsilon^n\right)$. Suppose $b^{n_k}\left(\underline{\theta}_\epsilon^{n_k}\right) = b^n\left(\bar{\theta}_\epsilon^{n_k}\right) = b_p^{n_k}$ for all n_k along a subsequence. Then, $\lim \Pr\left(P^{n_k} = b_p^{n_k}|V = v\right) \geq \epsilon > 0$ for each $v = 0, 1$ by Claim A.3. However, this means that there is pooling by pivotal types contradicting the assumption of the claim. □

Claim A.5. If the pivotal types are distinct and there is no pooling by pivotal types, then **H** aggregates information.

Proof. Fix any $\epsilon \in (0, 1/2)$. Claim A.4 implies $b^n(\underline{\theta}_\epsilon^n) < b_\epsilon^n < b^n(\bar{\theta}_\epsilon^n)$ for sufficiently large n. Given this definition, we have $\Pr\left(P^n \leq b_\epsilon^n|V = v\right) \leq \epsilon$ and $\lim \Pr\left(P^n \geq b_\epsilon^n|V = v'\right) \leq \epsilon$. Moreover,

$$\int_{p<b_\epsilon^n} \frac{\Pr\left(P^n = p|V = v\right)}{\Pr\left(P^n = p|V = v'\right)} \Pr\left(P^n = p|V = v'\right) dp = \int_{p<b_\epsilon^n} \Pr\left(P^n = p|V = v\right) dp \leq \epsilon$$

Therefore, $\Pr(P^n \in \{p < b_\epsilon^n : \frac{\Pr(V=v|P^n=p)}{\Pr(V=v'|P^n=p)} > \sqrt{\epsilon}\}|V = v') \leq \sqrt{\epsilon}$. Hence,

$$\lim \Pr(P^n \in \{\frac{\Pr\left(V = v|P^n = p\right)}{\Pr\left(V = v'|P^n = p\right)} > \sqrt{\epsilon}\}|V = v') \leq \\ \sqrt{\epsilon} + \lim \Pr\left(P^n \geq b_\epsilon^n|V = v'\right) < 2\sqrt{\epsilon}.$$

Finally, for any $\epsilon' > \sqrt{\epsilon}$ we find $\lim \Pr(P^n \in \{\frac{\Pr(V=v|P^n=p)}{\Pr(V=v'|P^n=p)} > \epsilon'\}|V = v') < 2\sqrt{\epsilon}$. Because, ϵ is arbitrary, we conclude that $\lim \Pr(P^n \in \{\frac{\Pr(V=v|P^n=p)}{\Pr(V=v'|P^n=p)} > \epsilon'\}|V =$

$v') = 0$ and a symmetric argument establishes the result for $V = v$. $\square$

$\square$

Lemma A.8. *If an equilibrium sequence aggregates information and* $\lim \mathbb{E}[P^n] > 0$*, then* P^n *converges in probability to* V*.*

Proof. We prove the result through two claims. In the first claim we show that if information is aggregated and the expected price is positive, then the pivotal types must be ordered. In the second claim we show that if the pivotal types are ordered, then price must converge to value.

Claim A.6. If $\mathbf{H}$ aggregates information and $\lim \mathbb{E}[P^n] > 0$, then $\sqrt{n}(F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)) \to \infty$.

Proof. If $\mathbf{H}$ aggregates information, then $\sqrt{n}\,|F_s^n(\theta_s^n(1)|1) - F_s^n(\theta_s^n(0)|1)| \to \infty$ and there is no pooling by pivotal types by Lemma 2.2. Pick a subsequence (abusing notation, we omit the relabeling of this subsequence) and assume, contrary to the claim that $\sqrt{n}\,(F_s^n(\theta_s^n(0)|1) - F_s^n(\theta_s^n(1)|1)) \to \infty$ along this subsequence. Moreover, suppose that $\lim \mathbb{E}[P^n|V = 0]$ and $\lim \mathbb{E}[P^n|V = 1]$ exist along this subsequence.

Recall the definition of b_ϵ^n given by Eq. A.3. The facts that $\mathbf{H}$ aggregates information and $\sqrt{n}\,(F_s^n(\theta_s^n(0)|1) - F_s^n(\theta_s^n(1)|1)) \to \infty$ together imply that

$$\lim \mathbb{E}[P^n|V = 0] \geq \lim \mathbb{E}[P^n] \geq \lim \mathbb{E}[P^n|V = 1]$$

and in particular $\lim \mathbb{E}[P^n|V = 0] \geq \lim \mathbb{E}[P^n] > 0$. This is because $\mathbb{E}[P^n|V = 0] \geq (1-\epsilon)b_\epsilon^n$ and $\mathbb{E}[P^n|V = 1\} \leq (1-\epsilon)b_\epsilon^n + \epsilon$ together imply that $\mathbb{E}[P^n|V = 0] + \epsilon \geq \mathbb{E}[P^n|V = 1]$ for each ϵ. Consider any type that submits a bid equal to b_ϵ^n. We have $\Pr(P^n < b_\epsilon^n|V = 1) \geq 1 - \epsilon$ and $\Pr(P^n > b_\epsilon^n|V = 0) \geq 1 - \epsilon$ by definition. Therefore, $u(b_\epsilon^n|\theta) \geq \Pr(V = 1|\theta)\,(1-\epsilon)\,(1 - \mathbb{E}[P^n|V = 1]) - \Pr(V = 0|\theta)\epsilon$ for any type θ. As ϵ is arbitrary, we find $\lim u(b^n(\theta)|\theta) \geq \Pr(V = 1|\theta)\,(1 - \lim \mathbb{E}[P^n|V = 1])$ for each θ.

For a given $\epsilon \in (0, \kappa_s)$, pick any type $\theta > \theta_s(0) \geq \theta_s(1)$ such that $\bar{F}_s^n(\theta|0) < \epsilon$. Note that $\lim \Pr(P^n \leq b^n(\theta)|V = v) = 1$ for $v = 0, 1$. This type wins with probability at least $\kappa_s - \epsilon$ in state $V = 0$. This is because if the type θ bids in a pool with $\theta_s(0)$, then the probability of winning is at least $\kappa_s - \epsilon$ in state $V = 0$ by Lemma A.2. Otherwise, this type wins with probability one in both states. Therefore, $\lim u(b^n(\theta)|\theta) \leq \Pr(V = 1|\theta)\,(1 - \lim \mathbb{E}[P^n|V = 1]) - (\kappa_s -$

$\epsilon)\Pr(V=0|\theta)\lim\mathbb{E}[P^n|V=0] < \Pr(V=1|\theta)\,(1-\lim\mathbb{E}[P^n|V=1])$ leading to a contradiction. □

Claim A.7. Suppose **H** aggregates information. If $\lim\mathbb{E}[P^n]>0$ or if $\sqrt{n}(F_s^n(\theta_s^n(1)|1)-F_s^n(\theta_s^n(0)|1))\to\infty$, then $\lim\mathbb{E}[P^n|V=0]=0$ and $\lim\mathbb{E}[P^n|V=1]=1$.

Proof. Information aggregation and $\lim\mathbb{E}[P^n]>0$ together imply that $\sqrt{n}(F_s^n(\theta_s^n(1)|1)-F_s^n(\theta_s^n(0)|1))\to\infty$ by the previous claim. Assume to the contrary that $\lim\mathbb{E}[P^n|V=0]>0$ along a convergent subsequence. There are two cases to consider: 1) There is an $\epsilon<\eta\lim\mathbb{E}[P^n|V=0]$ and a subsequence such that $\Pr(P^n=b^n(\bar{\theta}_\epsilon^n)|V=1)\to 0$; or alternatively 2) $\liminf\Pr(P^n=b^n(\bar{\theta}_\epsilon^n)|V=1)>0$ for all $\epsilon<\eta\lim\mathbb{E}[P^n|V=0]$ where $\bar{\theta}_\epsilon^n$ is the type defined in Eq. A.3.

Case 1: Our assumption that $\Pr(P^n=b^n(\bar{\theta}_\epsilon^n)|V=1)\to 0$ implies

$$\lim\Pr(P^n>b^n\left(\bar{\theta}_\epsilon^n\right)|V=1)=\lim\Pr\left(Y_s^n(k+1)>\bar{\theta}_\epsilon^n|V=1\right)=1-\epsilon.$$

Therefore,

$$\lim u(b^n\left(\bar{\theta}_\epsilon^n\right)|\bar{\theta}_\epsilon^n)\leq\lim\left(\Pr(V=1|\bar{\theta}_\epsilon^n)\epsilon-\Pr(V=0|\bar{\theta}_\epsilon^n)\mathbb{E}[P^n|V=0]\right).$$

However, $\lim\mathbb{E}[P^n|V=0]>0$ implies that $\lim u\left(b^n\left(\bar{\theta}_\epsilon^n\right)|\bar{\theta}_\epsilon^n\right)<0$ because $\epsilon<\eta\lim\mathbb{E}[P^n|V=0]$ and because $\frac{\Pr(V=0|\bar{\theta}_\epsilon^n)}{\Pr(V=1|\bar{\theta}_\epsilon^n)}=\frac{1}{l\left(\bar{\theta}_\epsilon^n\right)}>\eta$ leading to a contradiction. Therefore, $\lim\mathbb{E}[P^n|V=0]=0$.

Case 2: Our assumption $\liminf\Pr(P^n=b^n(\bar{\theta}_\epsilon^n)|V=1)>0$ implies

$$\lim\Pr(Y_s^n(k+1)\in\left\{\theta:b^n\left(\theta\right)=b^n\left(\bar{\theta}_\epsilon^n\right)\right\}|V=1)>0.$$

In other words, $\bar{\theta}_\epsilon^n$ bids in a pool and $\lim\sqrt{n}F_s^n(\{\theta:b^n\left(\theta\right)=b^n\left(\bar{\theta}_\epsilon^n\right)\}|V=1)>0$. However, such a pool is not possible if $\lim\Pr(P^n=b^n(\bar{\theta}_\epsilon^n)|V=1)>0$ and $\lim\Pr(P^n=b^n(\bar{\theta}_\epsilon^n)|V=0)=0$ by Lemma A.4.

Information aggregation and $\sqrt{n}(F_s^n(\theta_s^n(1)|0)-F_s^n(\theta_s^n(0)|0))\to\infty$ together imply that $\lim\Pr(P^n\leq b^n(\theta_s^n(0))|V=1)=0$. Therefore, $\lim u(b^n\left(\theta_s^n(0)\right)|\theta_s^n(0))=0$. However, $0=\lim u(b^n\left(\theta_s^n(0)\right)|\theta_s^n(0))\geq\lim u(b=1|\theta_s^n(0))=\lim\Pr(V=1|\theta_s^n(0))(1-\mathbb{E}[P^n|V=1])$, i.e., $\lim\mathbb{E}[P^n|V=1]=1$. □

□

The following lemma also provides conditions for information aggregation that we frequently use.

Lemma A.9. *Fix a sequence of bidding equilibria* $\mathbf{H}$*. If* $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$ *or if* $F_s(1|1) \geq F_s(1|0)$ *and* $F_s(1|1) > \kappa_s$*, then there is no pooling by pivotal types and price converges to value.*

Proof. For the following argument, note that $F_s(1|1) \geq F_s(1|0)$ and $F_s(1|1) > \kappa_s$ together imply that $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$ (see Lemma A.3). Under the lemma's assumptions the pivotal types are distinct and pooling by pivotal types is incompatible with equilibrium by Lemma A.3. However, then Lemma 2.2 implies that information is aggregated and Claim A.7 further implies that price converges to value because $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$. □

A.2. The Market Selection Lemmata. This section characterizes market selection. Throughout the section, we define $a_s^H(\theta) := a^H(\theta)$ and $a_r^H(\theta) := 1 - a^H(\theta)$ to simplify exposition.

Lemma A.10. *Suppose that* $a_m^H(\theta') > 0$ *for some type* θ' *in an equilibrium* H*. If* $u^H(m, b(\theta')|V = 0) < u^H(m', b|V = 0)$*, for* $m \neq m'$ *and some bid* $b \geq 0$*, then* $u(m, b(\theta')|\theta) > u(m', b|\theta)$ *for all* $\theta > \theta'$ *such that* $\theta \notin \mathcal{E}(\theta')$*.*

Proof. Fix an equilibrium H. For the remainder of the proof we suppress reference to the equilibrium H. Note that $u(m, b'|\theta, V = v) = u(m, b'|V = v)$ for any b', θ and v. Writing down the profit for type θ from bidding b in market m, we obtain $u(m, b|\theta) = u(m, b|V = 0)\Pr(V = 0|\theta) + u(m, b|V = 1)\Pr(V = 1|\theta)$. Our initial assumption that $a_m(\theta') > 0$ implies $u(m, b(\theta')|\theta') - u(m', b|\theta') \geq 0$. Moreover, $u(m, b(\theta')|\theta') - u(m', b|\theta') \geq 0$ and $u(m, b(\theta')|V = 0) < u(m', b|V = 0)$ together imply that $u(m, b(\theta')|V = 1) - u(m', b|V = 1) > 0$. Hence, if $\theta > \theta'$ and $\theta \notin \mathcal{E}(\theta')$, then

$$\begin{aligned}
&(u(m, b(\theta')|V = 0) - u(m', b|V = 0))\Pr(V = 0|\theta) + \\
&\qquad (u(m, b(\theta')|V = 1) - u(m', b|V = 1))\Pr(V = 1|\theta) > \\
&\qquad (u(m, b(\theta')|V = 0) - u(m', b|V = 0))\Pr(V = 0|\theta') + \\
&\qquad (u(m, b(\theta')|V = 1) - u(m', b|V = 1))\Pr(V = 1|\theta') \\
&\qquad\qquad = u(m, b(\theta')|\theta') - u(m', b|\theta') \geq 0
\end{aligned}$$

because $\Pr(V = 1|\theta) > \Pr(V = 1|\theta')$. □

Below we define $\hat{\theta}_m$ for $m \in \{s, r\}$ as the smallest type which wins a good with positive probability if $V = 0$ at the limit as n grows large, i.e., this type is the smallest "active" type in state $V = 0$.

Definition A.1. Fix a sequence of strategies $\{H^n\}$. If $F_m(1|0) \geq \kappa_m$, let $\theta_m^n(\epsilon) := \inf\{\theta : H^n([0,1] \times m \times (b^n(\theta), 1] | 0) < \kappa_m - \epsilon\}$, $\hat{\theta}_m(\epsilon) := \limsup \theta_m^n(\epsilon)$, and $\hat{\theta}_m := \inf_{\epsilon>0} \theta_m(\epsilon)$. If $F_m(1|0) < \kappa_m$, let $\hat{\theta}_m = \inf\{\theta : F_m(\theta|0) > 0\}$, and $\hat{\theta}_m = 1$ if the set is empty.

Suppose that $F_s(1|0) \geq \kappa_s$. The definition above selects type $\hat{\theta}_s = \theta_s(0)$ if the bidding function b^n is strictly increasing at $\theta_s^n(0)$ for sufficiently large n. The definition has more bite if, on the other hand, $\theta_s^n(0)$ submits a pooling bid. If $\theta_s^n(0)$ submits a pooling bid, then there are types $\underline{\theta}_p^n \leq \theta_s^n(0) \leq \theta_p^n$ who submit the same bid as $\theta_s^n(0)$. There are two cases to consider: In the first case $\theta_s(0) = \lim \theta_p^n$. Then the definition selects $\hat{\theta}_s = \theta_s(0)$. In the second case, if $\theta_s(0) < \lim \theta_p^n$, then the definition selects $\hat{\theta}_s = \lim \underline{\theta}_p^n$.

Lemma A.11. *Suppose that for an equilibrium sequence* $\mathbf{H}$ *we have that* $\lim \mathbb{E}(P_s^n|0) = 0$ *and* $\lim \mathbb{E}(P_r^n|0) > 0$*, then* $\lim a_r^n(\theta) = 1$ *for any* $\theta > \hat{\theta}_r$*.*

Proof. The fact that $\lim_n \mathbb{E}(P_s^n|V = 0) = 0$ implies that $\lim u^n(s, b|V = 0) = 0$ for any b. Pick an $\epsilon > 0$ and a sequence of types $\theta^n \in [\hat{\theta}_r^n(\epsilon/2), \hat{\theta}_r^n(\epsilon)]$ such that the limits $\lim \theta^n$, $\lim \hat{\theta}_r^n(\epsilon/2)$, $\lim \hat{\theta}_r^n(\epsilon)$ all exist and $a_r^n(\theta^n) > 0$. The probability that $P_r^n \leq b_r^n(\theta^n)$ converges to one in state 0. Therefore, the probability that θ^n wins an object in state 0 converges to one if this type does not bid in an atom along the sequence. Otherwise, the probability that this type wins is at least $\epsilon/2$ (see Lemma A.2 for this calculation). Hence, $\lim u(r, b_r^n(\theta^n)|V = 0) \leq -\frac{\epsilon}{2} \lim \mathbb{E}[P_r^n|V = 0] < 0 = \lim u^n(s, b|V = 0)$. Lemma A.10 then implies that $\lim u(r, b_r^n(\theta^n)|\theta) > \lim u(s, b|\theta)$ for any b and any type $\theta > \lim \theta^n$ such that $\theta \notin \mathcal{E}(\lim \theta^n)$ and therefore $a_r(\theta) = 1$. Similarly, if $\theta > \lim \theta^n$ and $\theta \in \mathcal{E}(\lim \theta^n)$, then $a_r^n(\theta) = 1$. This is because we can pick, without loss of generality, a pure and increasing representation of the market selection strategy a_r^n over $\mathcal{E}(\lim \theta^n)$. Since ϵ is arbitrary and $\hat{\theta}_r = \inf_\epsilon \hat{\theta}_r(\epsilon)$ we conclude that $\lim a_r^n(\theta) = 1$ for any $\theta > \hat{\theta}_r$. $\square$

Lemma A.12. *If* $\lim a_r^n(\theta) = 1$ *for all* $\theta > \theta_{en}$ *and* $\kappa_s > \bar{\kappa}$*, then either* $\theta_s(0) > \theta_s(1)$ *or* $\kappa_s > F_s(1|1)$*. Alternatively, if* $\lim a_r^n(\theta) = 0$ *for all* $\theta < \theta_{en}$ *and* $\kappa_s < \bar{\kappa}$*, then* $\theta_s(0) < \theta_s(1)$*.*

Proof. We argue that $\lim a_r^n(\theta) = 1$ for all $\theta > \theta_{en}$, $\kappa_s \leq F_s(1|1)$, and $\kappa_s > \bar{\kappa}$, together imply that $\theta_s(0) - \theta_s(1) > 0$. Let L_1 denote the set of measurable functions $\alpha : [0,1] \to [0,1]$ and consider the optimization problem

$$W(\kappa_s, \theta_{en}) = \max_{\alpha \in L_1} \frac{\int_0^{\theta_{en}} \alpha(\theta) dF(\theta|1)}{\int_0^{\theta_{en}} \alpha(\theta) dF(\theta|0)} \; s.t. \int_0^{\theta_{en}} \alpha(\theta) dF(\theta|1) = \kappa_s.$$

MLRP implies that $W(\kappa_s, \theta_{en}) = \frac{F([\theta', \theta_{en}]|1)}{F([\theta', \theta_{en}]|0)}$ where θ' is the type such that $F([\theta', \theta_{en}]|1) = \kappa_s$.[17] If $\kappa_s > \bar{\kappa}_{en} = F([\theta^*(\theta_{en}), \theta_{en}]|0) = F([\theta^*(\theta_{en}), \theta_{en}]|1)$, then $\theta' < \theta^*(\theta')$, and MLRP implies $F([\theta', \theta^*(\theta_{en})]|0) > F([\theta', \theta^*(\theta_{en})]|1)$. Therefore, $W(\kappa_s, \theta_{en}) < 1$.

Assume $\kappa_s > F_s(1|1)$ and define $\alpha^*(\theta)$ as the function which is equal to zero for all $\theta \le \theta_s(1)$ and equal to $a_s(\theta)$ for all $\theta > \theta_s(1)$. This function α^* is feasible for the above maximization problem. Therefore, we obtain $\frac{\bar{F}_s(\theta_s(1)|1)}{\bar{F}_s(\theta_s(1)|0)} = \frac{\int_{\theta_s(1)}^{\theta_{en}} a_s(\theta)dF(\theta|1)}{\int_{\theta_s(1)}^{\theta_{en}} a_s(\theta)dF(\theta|0)} = \frac{\int_0^{\theta_{en}} \alpha^*(\theta)dF(\theta|1)}{\int_0^{\theta_{en}} \alpha^*(\theta)dF(\theta|0)} \le W(\kappa_s, \theta_{en}) < 1$. Hence, $\theta_s(0) > \theta_s(1)$.

We now argue that if $\lim a_r^n(\theta) = 0$ for all $\theta < \theta_{en}$ and $\kappa_s < \bar{\kappa}$, then $\theta_s(0) < \theta_s(1)$. Define θ' as the type such that $F([\theta', \theta_{en}]|1) = \kappa_s$. Consider the following minimization problem $W(\kappa_s, \theta') = \min_{\alpha \in L_1} \frac{\int_{\theta'}^1 \alpha(\theta)dF(\theta|1)}{\int_{\theta''}^1 \alpha(\theta)dF(\theta|0)}$ $s.t.$ $\int_{\theta'}^1 \alpha(\theta)dF(\theta|1) = \kappa_s$. MLRP implies that $W(\kappa_s, \theta') = \frac{F([\theta', \theta_{en}]|1)}{F([\theta', \theta_{en}]|0)}$. Also, if $\kappa_s < \bar{\kappa} = F([\theta^*(\theta_{en}), \theta_{en}]|0) = F([\theta^*(\theta_{en}), \theta_{en}]|1)$, then $\theta' > \theta^*(\theta_{en})$, and hence $W(\kappa_s, \theta') > 1$ by MLRP. Define $\alpha^*(\theta)$ as the function that is equal to zero for all $\theta \le \theta_s(1)$ and equal to $a_s(\theta)$ for all $\theta > \theta_s(1)$. This α^* is feasible for the minimization problem. Therefore, $\frac{\bar{F}_s(\theta_s(1)|1)}{\bar{F}_s(\theta_s(1)|0)} = \frac{\int_{\theta_s(1)}^1 a_s(\theta)dF(\theta|1)}{\int_{\theta_s(1)}^1 a_s(\theta)dF(\theta|0)} = \frac{\int_{\theta'}^1 \alpha^*(\theta)dF(\theta|1)}{\int_{\theta'}^1 \alpha^*(\theta)dF(\theta|0)} \ge W(\kappa_s, \theta') > 1$ and hence $\theta_s(1) > \theta_s(0)$. □

A.3. Proof of Theorem 3.1. In the following lemma we characterize behavior in market r. We then use this lemma to prove Theorem 3.1.

Lemma A.13. *If $c > 0$, then $F_r(1|0) < F_r(1|1) \le \kappa_r$ along any equilibrium sequence. Moreover, the price in market r converges to c almost surely if $V = 0$ and converges to a random variable $P_r(1)$ if $V = 1$. The random variable $P_r(1)$ is equal to c with probability $q > 0$ and is equal to 1 with the remaining probability.*

Proof. The following three steps will together prove the result.

Step 1. $F_r(1|0) < F_r(1|1) \le \kappa_r$. We will argue that $F_r(1|0) < F_r(1|1)$. If $F_r(1|0) < F_r(1|1)$, then we must have $F_r(1|1) \le \kappa_r$. This is because $F_r(1|0) < F_r(1|1)$ and $F_r(1|1) > \kappa_r$ together imply that $P_r^n \to 1$ if $V = 1$ by Lemma A.9. But this is not possible because all the bidders in market r would then earn negative profits.

We now show $F_r(1|0) < F_r(1|1)$. First, suppose that $F_r(1|0) > F_r(1|1)$. This implies that $F_s(1|0) < F_s(1|1)$. There are two cases: $F_s(1|1) > \kappa_s$ and $F_s(1|1) \le \kappa_s$. If $F_s(1|1) > \kappa_s$, then $P_s^n \to 0$ if $V = 0$ by Lemma A.9, and if $F_s(1|1) \le \kappa_s$,

[17] In other words, the function $\alpha^*(\theta)$, which is equal to one if $\theta \ge \theta'$ and equal to zero otherwise is a maximizer of the problem.

then again $P_s^n \to 0$ if $V = 0$ because $F_s(1|0) < \kappa_s$. However, if $P_s^n \to 0$ when $V = 0$, then $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ by Lemma A.11. However, if $F_r(1|0) < \kappa_r$, then $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ implies that $F_r(1|1) > F_r(1|0)$, which contradicts our initial assumption. On the other hand, if $F_r(1|0) \geq \kappa_r$, then $F_r(1|1) > \kappa_r$ However, $F_r(1|1) > \kappa_r$ and $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ together imply that $P_r^n \to 1$ if $v = 1$ by Lemma A.9, which is not possible.

Second, suppose that $F_r(1|0) = F_r(1|1)$. There are two cases to consider: $F_r(1|1) > \kappa_r$ and $F_r(1|1) \leq \kappa_r$. If $F_r(1|1) > \kappa_r$, then $P_r^n \to 1$ if $V = 1$ by Lemma A.9, which is not possible. Alternatively, If $F_r(1|1) \leq \kappa_r$, then $F_s(1|1) > \kappa_s$. However, $F_s(1|0) = F_s(1|1)$ and $F_s(1|1) > \kappa_s$ together imply by Lemma A.9 that $P_s^n \to 0$ if $V = 0$. However, as argued previously, if $P_s^n \to 0$ if $V = 0$ and if $F_r(1|0) \leq \kappa_r$, then almost all types in market r win an object when $V = 0$ at a price which is at least c. Therefore, $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ by Lemma A.11. Thus, we conclude that $F_r(1|1) > F_r(1|0)$ because $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$. However, this contradicts that $F_r(1|0) = F_r(1|1)$, as we initially assumed.

Step 2. Assume $\lim \sqrt{n}\left(F_r^n(1|1) - \kappa_r\right) > -\infty$ –i.e., there are more bidders than objects in market r with positive probability in state 1. We have $b_r^n(\theta) \to 1$ for any type θ that bids in market r.

For any $\epsilon > 0$, pick θ^n such that $\Pr(Y_r^{n-1}(n\kappa_r) \in (0, \theta^n)|V = 1) \leq \epsilon$, and recall that that $Y_r^{n-1}(n\kappa_r) = 0$ if there are fewer than $n\kappa_r + 1$ bidders in market r. For sufficiently small ϵ, $\lim \Pr(Y_r^{n-1}(n\kappa_r) \geq \theta^n|V = 1) > 0$ because $\lim \Pr(Y_r^{n-1}(n\kappa_r) = 0|V = 1) < 1$ by assumption.

We argue that $\lim b_r^n(\theta) = 1$ for any $\theta > \lim \theta^n$. Any type θ^n in this sequence can ensure winning an object by submitting a bid equal to one in the auction. Therefore, $u(r, b_r^n(\theta^n)|\theta^n) = \mathbb{E}[V - P_r^n|b_r^n(\theta^n)\ win, \theta] \Pr(b_r^n(\theta^n)\ win|\theta) \geq u(r, b = 1|\theta^n)$. Noticing that,

$$
\begin{aligned}
u(r, b = 1|\theta^n) = \mathbb{E}[V - P_r^n|b_r^n(\theta^n)\ win, \theta] \Pr(b_r^n(\theta^n)\ win|\theta) + \\
\mathbb{E}[V - P_r^n|b_r^n(\theta^n)\ lose, \theta] \Pr(b_r^n(\theta^n)\ lose|\theta)
\end{aligned}
$$

we find $\mathbb{E}\left[V - P_r^n|b_r^n(\theta^n)\ lose, \theta\right] \Pr\left(b_r^n(\theta^n)\ lose|\theta\right) \leq 0$.

First, note that $\Pr(P_r^n \geq b_r^n(\theta^n)|V = 0) \leq e^{-\frac{\delta^2 n F_r^n(1|0)}{2+\delta}}$ by applying Chernoff's inequality (see Janson et al. (2011, Theorem 2.1)) where $\delta := \frac{\kappa_r}{F_r^n(1|0)} - 1$. Therefore, $\Pr(P_r^n \geq b_r^n(\theta^n)|V = 0) \leq e^{-\frac{\delta^2 n F_r^n(1|0)}{2+\delta}}$. Suppose that $\lim \Pr(P_r^n = b_r^n(\theta^n)|V = 1) = 0$. Then $\Pr(b_r^n(\theta^n)\ lose|V = 1) = \lim \Pr(P_r^n \geq b_r^n(\theta^n)|V = 1) > 0$. The fact that

$\mathbb{E}[V - P_r^n | P_r^n \geq b_r^n(\theta^n), \theta] \Pr(P_r^n \geq b_r^n(\theta^n)|\theta) \leq 0$ implies that

$$
\begin{aligned}
\lim(1 - \mathbb{E}[P_r^n | P_r^n \geq b_r^n(\theta^n), V = 1]) &\leq c \lim \frac{\Pr(P_r^n \geq b_r^n(\theta^n)|V = 0)}{\Pr(P_r^n \geq b_r^n(\theta^n)|V = 1) l(\theta^n)} \\
&= c \lim \frac{e^{-\frac{\delta^2 n F_r^n(1|0)}{2+\delta}}}{\Pr(P_r^n \geq b_r^n(\theta^n)|V = 1) l(\theta^n)} = 0,
\end{aligned}
$$

i.e., $\lim b_r^n(\theta) = 1$ for almost all $\theta > \lim \theta^n$. Alternatively, suppose that $\lim \Pr(P_r^n = b_r^n(\theta^n)|V = 1) > 0$. If $\lim \Pr(P_r^n = b_r^n(\theta^n)|V = 1) > 0$, then Lemma A.2 implies that there is a constant A such that $\Pr(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ lose|V = 1) \geq A/\sqrt{n}$ for all sufficiently large n. Therefore, $(1 - b_r^n(\theta^n)) \Pr(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ lose|V = 1) l(\theta^n) - c \Pr(b_r^n(\theta^n)\ lose|V = 0) \leq 0$, i.e., $\lim(1 - b_r^n(\theta^n)) \leq \lim \frac{c}{A}\sqrt{n} e^{-\frac{\delta^2 n F_r^n(1|0)}{2+\delta}} = 0$. Therefore, $\lim b_r^n(\theta) = 1$ for all $\theta \geq \lim \theta^n$.

Step 3 The price in market r converges to c almost surely if $V = 0$ and converges to a random variable $P_r(1)$ if $V = 1$. The random variable $P_r(1)$ is equal to c with probability $q > 0$ and is equal to 1 with the remaining probability.

The fact that the price converges to c almost surely if $V = 0$ follows from the LLN and the fact that $F_r(1|0) < \kappa_r$. Also, note that $\lim \sqrt{n}\,(F_r^n(1|1) - \kappa_r) < \infty$. This is because if $\lim \sqrt{n}\,(F_r^n(1|1) - \kappa_r) = \infty$, then the price clears at the bid of some type with probability one in state 1. However, the previous claim showed that $b_r^n(\theta) \to 1$ for all θ. But then $P_r(1) \to 1$, which implies that all bidders make a loss. The fact that $\lim \sqrt{n}\,(F_r^n(1|1) - \kappa_r) < \infty$ implies that $P_r(1)$ is equal to c with probability $q > 0$. With the remainder of the probability, i.e., with probability $1 - q$, the auction clears at the bid of some type θ and $b_r^n(\theta) \to 1$. Therefore, the auction price is equal to 1 with probability $1 - q$. □

Proof of Theorem 3.1. Fix an equilibrium sequence $\mathbf{H}$. If $c > 0$, then information is not aggregated in market r by Lemma A.13. We now prove the other assertions in the theorem.

If $c > 0$ and $\kappa_s > \bar{\kappa}$, then information is not aggregated in market s. Assume, on the way to a contradiction, that information is aggregated in market s. First suppose that $\lim_n \mathbb{E}[P_s^n] = 0$. Note that $\lim_n \mathbb{E}[P_r^n|V = 0] = c > 0$ by Lemma A.13. Therefore, if $\lim_n \mathbb{E}[P_s^n] = 0$, then all types would prefer to submit a bid equal to one in market s for all sufficiently large n. But if all types bid in auction s, then $F_s(\theta_s(1)|1) > F_s(\theta_s(0)|1)$ and Lemma A.9 implies that $\lim_n \mathbb{E}[P_s^n|V = 0] = 0$, $\lim_n \mathbb{E}[P_s^n|V = 1] = 1$ and therefore $\lim_n \mathbb{E}[P_s^n] = 1/2$ which contradicts that $\lim_n \mathbb{E}[P_s^n] = 0$. Hence, if information is aggregated in auction, then price converges to value by Lemma A.8.

If price converges to value in auction s, then $\lim u^n(s, b_s^n(\theta)|\theta) = 0$ for all θ. We first argue that $\lim a_r^n(\theta) = 1$ for all $\theta > \theta_{en}$. Recall that $\hat{\theta}_r$ is the smallest type that wins and object in state 0 in market r (definition A.1). If $\theta > \hat{\theta}_r$, then $\lim a_r^n(\theta) = 1$ by Lemma A.11 because $\lim_n \mathbb{E}[P_r^n|V = 0] > 0$ and $\lim_n \mathbb{E}[P_s^n|V = 0] = 0$. Also, note that $\hat{\theta}_r \leq \theta_{en}$ because if $\hat{\theta}_r > \theta_{en}$, then $\lim_n \mathbb{E}[P_r^n|V = 1] = c$ because $\theta_r^F(1) \leq \theta_{en}$ by Definition 3.1. However, if $\lim_n \mathbb{E}[P_r^n|V = 1] = c$, then $\lim u^n(r, b_r^n(\theta)|\theta) > 0$ for all $\theta \in (\theta_{en}, \hat{\theta}_r)$, contradicting that $\hat{\theta}_r \leq \theta_{en}$.

If $F_s(1|1) < \kappa_s$, then $P_s^n \to 0$ in state 1 showing that information is not aggregated in market s. Instead suppose that $F_s(1|1) \geq \kappa_s$. Lemma A.12 shows that if $\lim a_r^n(\theta) = 1$ for all $\theta > \theta_{en}$ and if $\kappa_s > \bar{\kappa}$, then $\theta_s^n(0) - \theta_s^n(1) \geq 0$ for all sufficiently large n. If $F_s(1|1) \geq \kappa_s$, then $\theta_s^n(0) - \theta_s^n(1) \geq 0$ for all sufficiently large n; however, this contradicts our initial assumption that information is aggregated in market s. This is because information aggregation in market s implies that $\theta_s^n(1) - \theta_s^n(0) > 0$ for all sufficiently large n.

If $c > 0$ and $\kappa_s < \bar{\kappa}$, then information is aggregated in market s. We prove this by looking at two cases. First, assume that

$$\theta_{en} = \inf\left\{\theta : \Pr\left(V = 1|\theta\right) > c\right\}.$$

The fact that $\kappa_s < \bar{\kappa}_{en}$ implies that $F_s\left(\theta_s(1)|1\right) > F_s\left(\theta_s(0)|1\right)$, even if all $\theta \geq \theta_{en}$ choose market r by Lemma A.12. If $F_s\left(\theta_s(1)|1\right) > F_s\left(\theta_s(0)|1\right)$, then Lemma A.9 implies that information is aggregated. Second, assume that $\theta_{en} = \theta_r^F(1)$. Lemma A.13 implies that $F_r\left(1|1\right) \leq \kappa_r$. However, if $F_r\left(1|1\right) \leq \kappa_r$, then Lemma A.12 implies that $F_s\left(\theta_s(1)|1\right) > F_s\left(\theta_s(0)|1\right)$. If $F_s\left(\theta_s(1)|1\right) > F_s\left(\theta_s(0)|1\right)$, then Lemma A.9 implies that information is aggregated. □

A.4. Analysis of the Illustrative Example. Here we construct equilibria under the assumptions of Section 4 for $c \neq 1/2$. As we noted earlier, no $\theta \in \mathcal{E}(0)$ bids in market r; if $c > 1/2$, then no $\theta \in \mathcal{E}(1/2)$ bids in market r; $b_m^n(\theta) = 0$ and $b_m^n(\theta) = 1$ for each $\theta \in \mathcal{E}(0)$ and $\theta \in \mathcal{E}(1)$, respectively, in any equilibrium. For each $\theta \in \mathcal{E}(1/2)$, let

$$b_m^n(\theta) := \frac{h\left(Y_m^{n-1}(k_m) = \theta|V = 1\right)}{h\left(Y_m^{n-1}(k_m) = \theta|V = 0\right)} / (1 + \frac{h\left(Y_m^{n-1}(k_m) = \theta|V = 1\right)}{h\left(Y_m^{n-1}(k_m) = \theta|V = 0\right)}). \tag{A.4}$$

where $h(Y_m^{n-1}(k_m) = \theta|V = 1) := \frac{d}{d\theta}\Pr(Y_m^{n-1}(k_m) \leq \theta|V = 1)$, i.e., h is a binomial density.

Proposition A.1. *An equilibrium exists for each n and in any equilibrium all* *types θ bid $b^n_m(\theta)$. If $1-g>\kappa_r$, then information is aggregated in both markets. If $1-g<\kappa_r$, then all equilibrium sequences converge to the following outcome: If $V=0$, then $P^n_r\to c$ with probability one. If $V=1$, then P^n_r converges to a random variable P_r that is equal to c with probability $c/(1-c)$ and equal to one with the remaining probability. In market s, P^n_s converges in distribution to a random variable P_s and $\Pr[P_s\le p|V=v]$ is atomless, increasing on $[0,1]$ for $v=0,1$. If $c>1/2$, then $\mathbb{E}[P_r|V=1]=\mathbb{E}[P_r|V=0]=c$, $\mathbb{E}[P_s|V=1]=c$, and $\mathbb{E}[P_s|V=0]=1-c$. If $c<1/2$, then $\mathbb{E}[P_r|V=1]=\mathbb{E}[P_s|V=1]=1-c$, and $\mathbb{E}[P_r|V=0]=\mathbb{E}[P_s|V=0]=c$.*

Proof. **Step 1.** All types $\theta\in\mathcal{E}(1/2)$ bid $b^n_m(\theta)$ in any bidding equilibrium.

If $F^n_m(\mathcal{E}(1)|1)>0$, then the bidding distribution has no atoms except at $b=1$ and $b=0$ and therefore $b^n_m(\theta)$ is given by Eq. (A.4) for each $\theta\in\mathcal{E}(1/2)$ by Lemma 2.1. To see this, define an auxiliary type distribution G where all types that choose market $m'\ne m$ are assigned signal $\theta\in\mathcal{E}(0)$, i.e., $G(\mathcal{E}(1/2)|v)=F_m(\mathcal{E}(1/2)|v)$, $G(\mathcal{E}(1)|v)=F_m(\mathcal{E}(1)|v)$ and $G(\mathcal{E}(0)|v)=1-G(\mathcal{E}(1/2)|v)-G(\mathcal{E}(1)|v)$. Bidding in market m under G is the same as F_m except at $b=0$ and G satisfies MLRP. Therefore, no type $\theta\in\mathcal{E}\,(1/2)$ can bid in an atom because otherwise we have a contradiction to Lemma 7 in Pesendorfer and Swinkels (1997). If $F^n_m(\mathcal{E}(1)|1)=0$, then Eq. (A.4) implies $b^n_m(\theta)=1/2$ for each $\theta\in\mathcal{E}(1/2)$. Any type $\theta\in\mathcal{E}(1/2)$ would always under cut any atom $b>1/2$ and out bid any atom $b<1/2$. Therefore, types $\theta\in\mathcal{E}(1/2)$ can bid in an atom only at $b=1/2$. If the bid distribution is increasing over an interval, these types bid $b^n_m(\theta)=1/2$ by Lemma 2.1. Therefore, all types $\theta\in\mathcal{E}(1/2)$ bid $1/2$ as required.

Step 2. There exists $\theta_1\in\mathcal{E}(1/2)$, $\theta_2\in\mathcal{E}(1)$ and an equilibrium where all types $\theta\in[0,\theta_1)\cup(2/3,\theta_2]$ bid $b^n_s(\theta)$ in market s and all others bid $b^n_r(\theta)$ in market r. The proof, which uses Kakutani's fixed point theorem, is in the online appendix.

Step 3. We have $u^n(\theta)\to0$ for any $\theta\in\mathcal{E}(1/2)$.

Pick m such that $\lim F^n_m(\mathcal{E}(1)|1)+F^n_m(\mathcal{E}(1/2)|1)>\kappa_m$. If $F^n_m(\mathcal{E}(1)|1)>0$ for all sufficiently large n, then $b^n_m(\cdot)$ is increasing in $\theta\in\mathcal{E}(1/2)$. Therefore, there exists a $\theta'\in\mathcal{E}(1/2)$ that bids in market m and wins an object in state $V=1$ with probability converging to zero. This type's equilibrium payoff $u^n(\theta')\to0$. Consequently, $u^n(\theta)=u^n(\theta')\to0$ for any $\theta\in\mathcal{E}(1/2)$. Alternatively, if $F^n_m(\mathcal{E}(1)|1)=0$ for all large n, then all $\theta\in\mathcal{E}(1/2)$ bid $1/2$ and hence $u^n(\theta)\to0$ for any $\theta\in\mathcal{E}(1/2)$.

Step 4. Suppose $\sqrt{n}\bar{F}^n_m(\mathcal{E}\,(1)\,|V=1)/\sigma\to x\in[0,\infty]$ and there is a sequence $\{\theta^n\}\subset\mathcal{E}\,(1/2)$ with $\lim\sqrt{n}(\kappa_m-\bar{F}^n_m(\theta^n|V=1))/\sigma=y$, where $\sigma=\sqrt{\kappa_m\,(1-\kappa_m)}$.

If $x < \infty$, then $\lim b_m^n(\theta^n) = \frac{\phi(y)}{\phi(y+x)}/(1 + \frac{\phi(y)}{\phi(y+x)})$, $\lim \Pr(P_m^n \leq b_m^n(\theta^n)|V = 1) = \Phi(y)$ and $\lim \Pr(P_m^n \leq b_m^n(\theta^n)|V = 0) = \Phi(y + x)$ by Lemma A.6. If $x = \infty$, then $b_m^n(\theta^n) \to 1$ because

$$\frac{\sqrt{n}h\left(Y_m^{n-1}(k_m) = \theta^n|V = 1\right)}{\sqrt{n}h\left(Y_m^{n-1}(k_m) = \theta^n|V = 0\right)} \to \frac{\phi(y)}{\sqrt{n}h\left(Y_m^{n-1}(k_m) = \theta^n|V = 0\right)} = \infty$$

since $\sqrt{n}h(Y_m^{n-1}(k_m) = \theta^n|V = 0) \to 0$.

Step 5. If $c > 1/2$, then $F_s^n(\mathcal{E}(1)|1) > 0$. If we further assume $1 - g < \kappa_r$, then $\lim \sqrt{n}F_s^n(\mathcal{E}(1)|1) < \infty$ and $P_r^n \to c$ in both states.

First we show $F_s^n(\mathcal{E}(1)|1) > 0$. If $F_s^n(\mathcal{E}(1)|1) = 0$, then $b_s^n(\theta) = 1/2$ for any $\theta \in \mathcal{E}(1/2)$. However then any type $\theta \in \mathcal{E}(1)$ can get an object from auction s with probability one at a price not more than $1/2$. Therefore, if $F_s^n(\mathcal{E}(1)|1) = 0$, then the payoff from participating in auction s is strictly greater than bidding in market r for $\theta \in \mathcal{E}(1)$ and this contradicts $F_s^n(\mathcal{E}(1)|1) = 0$.

If $1-g < \kappa_r$, then $P_r^n \to c$ in both states because $\lim F_r^n(\mathcal{E}(1)|1) \leq 1-g < \kappa_r$ and $F_r^n(\mathcal{E}(1/2)|1) = 0$. Now we show that $1-g < \kappa_r$ implies $\lim \sqrt{n}F_s^n(\mathcal{E}(1)|1) < \infty$. Assume to the contrary. Step 4, $\lim F_s^n(\mathcal{E}(1/2)|1) = g > \kappa_s$, and $\lim \sqrt{n}F_s^n(\mathcal{E}(1)|1) = \infty$ together imply that $P_s^n \to 1$ in state $V = 1$. However, then no type $\theta \in \mathcal{E}(1)$ would choose market s for sufficiently large n because $P_r^n \to c$ leading to a contradiction.

Step 6. If $c < 1/2$, then $F_s^n(\mathcal{E}(1)|1) > 0$ for all sufficiently large n. If we further assume $1 - g < \kappa_r$, then $\lim \sqrt{n}F_s^n(\mathcal{E}(1)|1) < \infty$, $P_r^n \to c$ in state $V = 0$, and $\lim \mathbb{E}[P_r^n|V = 1] = 1 - c$.

We first show if $1 - g < \kappa_r$ and $\lim F_s^n(\mathcal{E}(1)|1) = 0$, then $\lim F_r^n(\mathcal{E}(1/2)|1) = \kappa_r + g - 1$, $P_r^n \to c$ in state $V = 0$, and $\lim \mathbb{E}[P_r^n|V = 1] = 1 - c$. Subsequently, we show that if $1 - g < \kappa_r$, then $F_s^n(\mathcal{E}(1)|1) > 0$ for all sufficiently large n and $\lim \sqrt{n}F_s^n(\mathcal{E}(1)|1) < \infty$.

First, suppose that $\lim F_r^n(\mathcal{E}(1/2)|1) < \kappa_r + g - 1$. Then the price in market r converges to c in both states and each $\theta \in \mathcal{E}(1/2)$ that bids in market r wins an object with probability one at a price converging to c. However, then $\lim u^n(\theta) = 1/2 - c > 0$ for any such type but this contradicts Step 3. Suppose instead that $\lim F_r^n(\mathcal{E}(1/2)|1) > \kappa_r + g - 1$. Then the price in market r converges to one in state $V = 1$ by Step 4. But this would imply that the profit of any $\theta \in \mathcal{E}(1/2)$ that bids in market r is negative again leading to a contradiction. Hence, $\lim F_r^n(\mathcal{E}(1/2)|1) = \kappa_r + g - 1$ which further implies $P_r^n \to c$ in state $V = 0$ because $\lim F_r^n(\mathcal{E}(1/2)|0) < \kappa_r$ and $F_r^n(\mathcal{E}(1)|0) = 0$. Since the profit for any $\theta \in \mathcal{E}(1/2)$ from bidding in market r converges to zero by step 3 we further establish that $\lim \mathbb{E}\left[P_r^n|V=1\right] = 1-c$.

We next show $F_s^n(\mathcal{E}(1)|1) > 0$ for all sufficiently large n. On the way to a contradiction assume that $F_s^n(\mathcal{E}(1)|1) = 0$ for all sufficiently large n. This implies that $\lim \mathbb{E}\left[P_s^n|V=1\right] \geq \lim \mathbb{E}\left[P_r^n|V=1\right]$ because otherwise any type $\theta \in \mathcal{E}(1)$ would prefer to bid in market s instead of r for all sufficiently large n. If $1-g > \kappa_r$ and $F_s^n(\mathcal{E}(1)|1) = 0$, then $P_r \to 1$ and $P_s \to 1/2$. But this contradicts $\lim \mathbb{E}\left[P_s^n|V=1\right] \geq \lim \mathbb{E}\left[P_r^n|V=1\right]$. On the other hand if $1-g < \kappa_r$ and $F_s^n(\mathcal{E}(1)|1) = 0$ for all sufficiently large n, then $b_s^n(\theta) = 1/2$ for $\theta \in \mathcal{E}(1/2)$ for all sufficiently large n. Hence, $\lim \mathbb{E}\left[P_s^n|V=1\right] \leq 1/2 < \lim \mathbb{E}\left[P_r^n|V=1\right] = 1-c$ again leading to a contradiction.

We now show that if $1-g < \kappa_r$, then $\lim \sqrt{n} F_s^n\left(\mathcal{E}(1)|1\right) < \infty$. Assume $\lim \sqrt{n} F_s^n\left(\mathcal{E}(1)|1\right) = \infty$. If $F_s^n\left(\mathcal{E}(1)|1\right) + F_s^n\left(\mathcal{E}(1/2)|1\right) > \kappa_s$, then $P_s^n \to 1$ in state $V=1$ by Step 4. However, $\lim \mathbb{E}\left[P_r^n|V=1\right] \leq 1-c < 1$, which implies that no $\theta \in \mathcal{E}(1)$ would bid in market s for sufficiently large n, contradicting $\lim \sqrt{n} F_s^n\left(\mathcal{E}(1)|1\right) = \infty$. On the other hand, if $F_s^n\left(\mathcal{E}(1)|1\right) + F_s^n\left(\mathcal{E}(1/2)|1\right) \leq \kappa_s$, then $P_s^n \to 0$ in state $V=0$ by Step 4. The fact that $\theta \in \mathcal{E}(1)$ bids in market s implies that $\lim \mathbb{E}\left[P_s^n|V=1\right] \leq \lim \mathbb{E}\left[P_r^n|V=1\right] \leq 1-c$. But then bidding one in market s gives any $\theta \in \mathcal{E}(1/2)$ positive profit at the limit because $P_s^n \to 0$ and $\lim \mathbb{E}\left[P_s^n|V=1\right] \leq 1-c$ leading to a contradiction.

Step 7. If $1-g < \kappa_r$, then $P_s^n \to P_s$ in distribution and $\Pr(P_s \leq p|V=v)$ is atomless and increasing on $[0,1]$ for $v=0,1$. If $c > 1/2$, then $\lim \mathbb{E}\left[P_s^n|V=1\right] = c$ and $\lim \mathbb{E}\left[P_s^n|V=0\right] = 1-c$. If $c < 1/2$, then $\lim \mathbb{E}\left[P_s^n|V=1\right] = 1-c$ and $\lim \mathbb{E}\left[P_s^n|V=0\right] = c$.

If $1-g < \kappa_r$, then $F_s^n(\mathcal{E}\left(1/2\right)|1) > \kappa_s$ and $\sqrt{n}F_s^n(\mathcal{E}\left(1\right)|1)/\sigma = x < \infty$ by steps 5 and 6. For any y, pick $\theta^n \in \mathcal{E}\left(1/2\right)$ such that $\sqrt{n}\left(\kappa_s - \bar{F}_s^n(\theta^n|V=1)\right)/\sigma = y$. This type's bid is given by $b_s^n\left(\theta^n\right) \to \frac{\phi(y)}{\phi(y+x)}/(1+\frac{\phi(y)}{\phi(y+x)}) = e^{yx+\frac{x^2}{2}}/(1+e^{yx+\frac{x^2}{2}}) \in (0,1)$ by Step 4. Solving for y as a function of p using $p = e^{yx+\frac{x^2}{2}}/(1+e^{yx+\frac{x^2}{2}})$ we find $y = \frac{1}{x}\left(\ln\frac{p}{1-p} - \frac{x^2}{2}\right)$. Therefore, $\lim \Pr(P \leq p|V=1) = \Phi((\ln\frac{p}{1-p} - x^2/2)/x)$ and $\lim \Pr(P \leq p|V=0) = \Phi((\ln\frac{p}{1-p} + x^2/2)/x)$.

For type $\theta' \in \mathcal{E}\left(1/2\right)$, which wins an object in market s with probability converging to one in both states, we find $\lim u^n(s, b_s^n(\theta')|\theta') = (1 - \lim \mathbb{E}[P_s^n|V=1] - \lim \mathbb{E}[P_s^n|V=0])/2 = 0$. Therefore, $1 - \lim \mathbb{E}[P_s^n|V=1] = \lim \mathbb{E}[P_s^n|V=0]$. If $c > 1/2$, then we must have $\lim \mathbb{E}\left[P_s^n|V=1\right] = c$ in order for types $\theta \in \mathcal{E}\left(1\right)$ to be indifferent between the two markets. If $c < 1/2$, then we must have $\lim \mathbb{E}\left[P_s^n|V=1\right] = \lim \mathbb{E}\left[P_r^n|V=1\right]$ and $\lim \mathbb{E}\left[P_s^n|V=0\right] = \lim \mathbb{E}\left[P_r^n|V=0\right] = c$

for for types $\theta \in \mathcal{E}(1)$ and types $\theta \in \mathcal{E}(1/2)$ to be indifferent between the two markets. Note that if $x = 0$, then $\lim \mathbb{E}[P_s^n|V = v] = 1/2$ for $v = 0, 1$ and hence we find $x > 0$ because $c \neq 1/2$.

Step 8. If $1 - g > \kappa_r$, then $P_s^n \to V$ and $P_r^n \to V + c(1 - V)$.

We show $\lim \mathbb{E}[P_m^n|V = 1] = 1$ for $m \in \{r, s\}$, implies that $\lim \mathbb{E}[P_s^n|V = 0] = 0$ and $\lim \mathbb{E}[P_r^n|V = 0] = c$. If $\lim \mathbb{E}[P_r^n|V = 1] = 1$, then $\lim F_r^n(\mathcal{E}(1/2)|1) = 0$ and therefore $\lim \mathbb{E}[P_r^n|V = 0] = c$. Moreover, $\mathbb{E}[P_s^n|V = 0] = 0$ because otherwise any type $\theta' \in \mathcal{E}(1/2)$, which wins an object in state $V = 0$ in market s with probability converging to one, would receive a negative payoff.

We now show that $\lim \mathbb{E}[P_m^n|V = 1] = 1$ for $m \in \{r, s\}$. Steps 5 and 6 establish that $F_s^n(\mathcal{E}(1)|1) > 0$ for all sufficiently large n and thus $\lim \mathbb{E}[P_s^n|V = 1] \leq \lim \mathbb{E}[P_r^n|V = 1]$. Moreover, if $F_s^n(\mathcal{E}(1)|1) \to 0$, then $F_r^n(\mathcal{E}(1)|1) \to 1 - g > \kappa_r$ and $\lim \mathbb{E}[P_s^n|V = 1] = \lim \mathbb{E}[P_r^n|V = 1]$. Since $b_r^n(\theta) = 1$ for $\theta \in \mathcal{E}(1)$, we find $\lim \mathbb{E}[P_r^n|V = 1] = 1$ and therefore $\lim \mathbb{E}[P_s^n|V = 1] = 1$. Alternatively, suppose $\lim F_s^n(\mathcal{E}(1)|1) > 0$. If $\lim(F_s^n(\mathcal{E}(1)|1) + F_s^n(\mathcal{E}(1/2)|1)) > \kappa_s$, then $\lim \mathbb{E}[P_s^n|V = 1] = 1$ by Step 4 and therefore $\lim \mathbb{E}[P_r^n|V = 1] = 1$. If $\lim(F_s^n(\mathcal{E}(1)|1) + F_s^n(\mathcal{E}(1/2)|1)) \leq \kappa_s$, then $\lim \mathbb{E}[P_s^n|V = 0] = 0$ because $\lim F_s^n(\mathcal{E}(1/2)|0) < \kappa_s$. However, if $\lim \mathbb{E}[P_s^n|V = 0] = 0$, then $\lim \mathbb{E}[P_s^n|V = 1] = 1$ because otherwise types in $\mathcal{E}(1/2)$ make positive profit in market s contradicting Step 3. □

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