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Reputation in Long-Run Relationships

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We model a long-run relationship as an infinitely repeated game played by two equally patient agents. In each period, the agents play an extensive-form stage game of perfect information with either locally non-conflicting interests or strictly conflicting interests. There is incomplete information about the type of Player 1, while Player 2's type is commonly known. We show that a sufficiently patient Player 1 can leverage Player 2's uncertainty about his type to secure his highest pay-off, compatible with Player 2's individual rationality, in any perfect Bayesian equilibrium of the repeated game.

Key words: Repeated games, Reputation, Equal discount factor, Long-run players

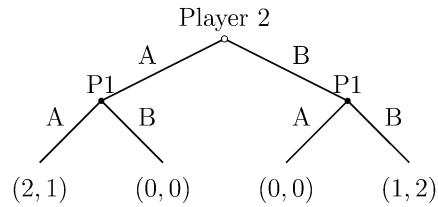
JEL Codes: C73, D83

1. INTRODUCTION

Maintaining a reputation can benefit economic agents since it lends credibility to their future commitments, threats, or promises. A reputation can help a firm commit to fight potential entrants (Kreps and Wilson, 1982; Milgrom and Roberts, 1982) or it can lend credibility to a government's monetary and fiscal policies (Barro, 1986; Phelan, 2006). So a patient agent may forego short-run profit in order to cultivate a reputation in anticipation of long-run benefits.

Reputation effects are pronounced if an agent is patient, *i.e*. if the short-run cost of building a reputation is less important to the agent than the long-run benefit (Fudenberg and Levine, 1989, 1992). There is a tension, however, if the agent trying to build a reputation faces an opponent who is equally patient: the opponent may also sacrifice pay-off in the short-run in order to extensively test the agent's resolve to go through with his commitments, threats, or promises. This can make it prohibitively expensive to build a reputation in a certain class of repeated games where players move simultaneously (Cripps and Thomas, 1997). To highlight this tension, we focus in this paper on equally patient agents and show that in repeated games where players move *sequentially* reputation effects are nevertheless prominent.

We consider an infinitely repeated game played by two *equally patient* agents. We assume that Player 2 is uncertain about the type of Player 1, while Player 1 is perfectly informed about the type of Player 2. In each period, the agents play an extensive-form game of *perfect information*. There are either *locally non-conflicting interests* (LNCI) or *strictly conflicting*



Note: The first component of the payoff vector is player 1's payoff.

FIGURE 1
The battle-of-the-sexes game

interests (SCI) in the stage game.[1] Within this framework, we prove a *reputation result*: a sufficiently patient Player 1 can guarantee his highest pay-off compatible with Player 2's individual rationality, in any perfect Bayesian equilibrium of the repeated game.

To make the discussion more concrete, consider the following strategic situation faced by a husband and wife, two legislators, or two countries: In each period of a long-run relationship, the two players must decide whether to undertake Player 1's preferred policy A, Player 2's preferred policy B, or neither of the two policies. Unanimity is required for any policy to be chosen. These policies can represent competing treaties in a pollution abatement negotiation between two countries, budget alternatives under consideration by two political rivals, or even weekend plans being bargained over by a married couple. The repeated game where Figure 1 is played in each period is a simple representation of this strategic situation.[2] Suppose that Player 2 (she) believes that Player 1 (he) is either fully rational or a Stackelberg type who is committed to choosing A in each period. A rational Player 1, cognizant of Player 2's uncertainty, has an incentive to mimic the Stackelberg type, *i.e.* to build a reputation. If Player 2 is convinced that Player 1 is the Stackelberg type, then she will have no choice but to play A, and policy A will be the outcome in each period. Therefore, a patient Player 1 may play A for many periods, even if Player 2 plays B (*i.e.* at the expense of reaching an agreement) in order to convince Player 2 that he is indeed the Stackelberg type. However, Player 2 knows that Player 1 has an incentive to mimic the Stackelberg type. Consequently, an equally patient Player 2 may play B (*i.e.* resist playing A) for many periods, thereby making reputation building particularly costly, especially if she deems it sufficient likely that Player 1 is rational and will eventually start playing B.

Given these two opposing forces, can Player 1 build a reputation and ensure that policy A is implemented? Or alternatively, will screening by Player 2 keep a rational Player 1 from building a reputation? These questions are addressed in our main finding: if the players are equally and arbitrarily patient, then policy A is implemented in each period and Player 1 receives a pay-off equal to two in any perfect Bayesian equilibrium of the repeated game. This outcome is

1. There are LNCI in a game if the unique pay-off profile where Player 1 receives his highest pay-off is strictly individually rational for Player 2. Intuitively, there are SCI in a game if the action which is the best for Player 1 is the worst for his opponent. See Assumption 1 for precise statements.

2. The battle-of-the-sexes game is used to model product compatibility in Farrell and Saloner (1988), network externalities in Katz and Shapiro (1985), communication and mediation in Banks and Calvert (1992), and repeated bargaining in Schelling (1960). For the battle-of-the-sexes game applied to pollution abatement negotiations between nations, see Harstad (2007); for an application to negotiations between political rivals, see Alesina and Drazen (1991); and for an application to marital bargaining, see Lundberg and Pollak (1994).

independent of which player moves first and independent of how small the initial uncertainty about Player 1's type is.[3]

In the previous example, Player 1's reputation allowed him to credibly commit to always choosing the same action. However, we can conceive of other strategic situations where Player 1 may want to commit to a more complex strategy that rewards or punishes his opponent in a history-dependent way. For example, Player 1 may want to be known for playing tit-for-tat or for punishing bad behaviour consistently. To capture reputation effects more generally, we assume that Player 1 is either fully rational or one of many *commitment* types. Each commitment type is programmed to play a certain repeated game strategy. The commitment type central to our analysis is a dynamic *Stackelberg* type. This type plays the repeated game strategy that Player 1 would choose if Player 1 could publicly commit to any repeated game strategy. Ideally, Player 1 would like to convince his opponent that his future actions will fully conform to the behaviour of the Stackelberg type. We show that a sufficiently patient Player 1 can use his ability to mimic the Stackelberg type and his opponent's uncertainty about his type to secure his most preferred outcome for the repeated game.

1.1. *Related literature*

This paper is closely related to the literature on reputation effects in repeated games. Much of the previous literature on reputation considers a patient Player 1 who faces a myopic opponent. Most prominently, Fudenberg and Levine (1989, 1992) show that if there is positive probability that Player 1 is a type committed to playing the Stackelberg action in every period, then Player 1 gets at least his static Stackelberg pay-off in any equilibrium of the repeated game.[4] Reputation results have also been established for repeated games where Player 1 faces a non-myopic opponent, but one who is sufficiently less patient than Player 1 (see Schmidt, 1993; Aoyagi, 1996; Celentani *et al*., 1996; or Evans and Thomas, 1997). However, the repeated games that these papers consider are genuinely long-run only for Player 1 and this feature is crucial for the reputation results.

In a game with a non-myopic opponent, Player 1 may achieve a pay-off that exceeds his static Stackelberg pay-off by using a history-dependent strategy that rewards or punishes Player 2. Conversely, future punishments or rewards can induce Player 2 to not best respond to a Stackelberg action and thereby force Player 1 below his static Stackelberg pay-off.[5] These complications render reputation effects fragile in repeated games with equally patient players: A reputation result obtains in a repeated simultaneous-move game only if the stage game is a strictly dominant action game (Chan, 2000) or if there are SCI in the stage game (Cripps, Dekel and Pesendorfer, 2005).[6] For other repeated simultaneous-move games, any individually rational pay-off can be sustained in a perfect equilibrium if the players are sufficiently patient (see the folk theorem of Cripps and Thomas, 1997).[7]

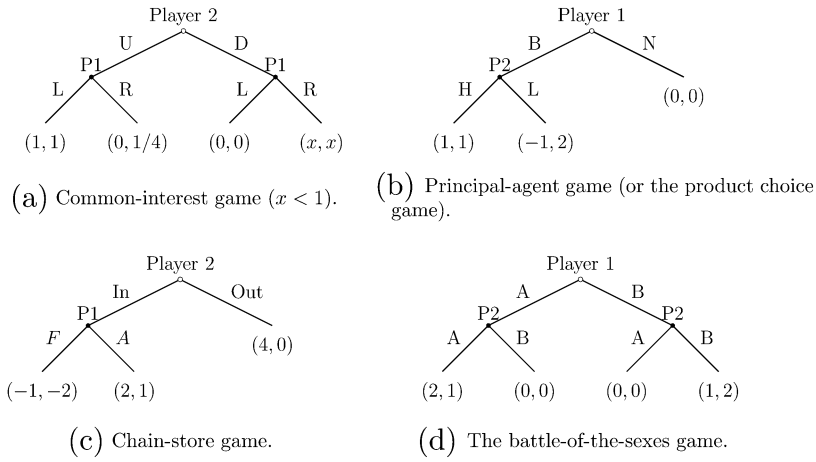
3. See Figure 2(d) for the battle-of-the-sexes game where Player 1 moves first.

4. The static Stackelberg pay-off for Player 1 is the highest pay-off he can guarantee in the stage game through public commitment to a stage game action (a Stackelberg action). See Mailath and Samuelson (2006, p. 465) for a formal definition.

5. Player 2 may expect punishments or rewards either from the rational type of Player 1 after he chooses a move that would not be chosen by the Stackelberg type (Celentani *et al*., 1996, Section 5, or Cripps and Thomas, 1997) or from a commitment type other than the Stackelberg type (Schmidt, 1993, or Celentani *et al*., 1996).

6. A stage game is a strictly dominant action game if Player 1's static Stackelberg pay-off is equal to his highest pay-off compatible with Player 2's individual rationality, and if the Stackelberg action is strictly dominant, see Mailath and Samuelson (2006, p. 540) for a formal definition.

7. Also, see Cripps and Thomas (1995) for a model of equally patient agents, which uses the limit of means criteria instead of equal discounting.



(a) Common-interest game ($x < 1$).

(b) Principal-agent game (or the product choice game).

(c) Chain-store game.

(d) The battle-of-the-sexes game.

FIGURE 2
Sequential-move games with (2a and 2b) LNCI or (2c and 2d) SCI

1.2. *Contribution to the literature*

We make three main contributions to the literature on reputation effects in repeated games with equally patient players. First, we provide a reputation result for a new class of games: repeated extensive-form games of perfect information. Second, we highlight the distinct role that perfect information plays for a reputation result with equally patient agents. Third, we introduce novel methods, inspired by the bargaining literature (Myerson, 1991, Chapter 8.8), to analyse reputation effects in repeated games.

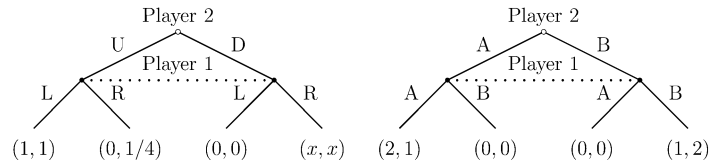
Previous reputation results for equally patient agents are for certain repeated simultaneous-move games (*i.e.* Chan, 2000, for strictly dominant action stage games and Cripps, Dekel and Pesendorfer, 2005, for stage games with SCI). In contrast, we focus on repeated extensive-form games of perfect information, and as our first main contribution, we establish a reputation result for stage games with LNCI or SCI.[8] Games that are commonly used in economic applications, such as the examples depicted in Figures 1 and 2, are included in the class of games that we cover in our reputation result.[9,10]

Games with LNCI have a common value component, whereas games with SCI entail conflict between the two players. A game has LNCI if the unique pay-off profile where Player 1 receives his highest stage game pay-off is strictly individually rational for Player 2. The battle-of-the-sexes game where Player 2 moves first (Figure 1), the common interest game (Figure 2(a)), and the principal–agent game (Figure 2(b)) have LNCI. These games have LNCI because Player 1 receives his highest pay-off in the pay-off profile (2, 1), (1, 1), and (1, 1), in Figures 1, 2(a), and

8. If a game in the class that we consider (*i.e.* a game with LNCI or SCI) is played under complete information, then the folk theorem of Fudenberg and Maskin (1986) applies under a full dimensionality condition (see Wen, 2002).

9. Examples of common interest games include the coordination game and the hawk-dove game (for applications, see Morris and Shin, 1998, and Baliga and Sjöström, 2004). For an application of the repeated principal–agent game, see Laffont and Martimort (2002, Chapter 9). For an application of the chain-store game in industrial organization, see Tirole (1988, Chapter 9).

10. A game falls outside of the class of games with LNCI or SCI if the profile where Player 1 receives his highest pay-off is not strictly individually rational for Player 2 and the game does not have SCI. Examples of such games include the prisoner's dilemma game and the principal–agent game in Figure 2(b) if Player 2 is the player that is building a reputation instead of Player 1. See Section 4.2 for a more extensive discussion.



(a) Common-interest game ($x < 1$). (b) The battle-of-the-sexes game.

FIGURE 3
Simultaneous-move games with LNCI

2(b), respectively. Moreover, in each of these profiles, Player 2's pay-off strictly exceeds her minimax, which is equal to zero, 1/4, and zero, in Figures 1, 2(a), and 2(b), respectively.

A game has SCI if Player 1 has an action (a Stackelberg action) such that any best reply to this action yields Player 1 his highest pay-off compatible with Player 2's individual rationality and yields Player 2 her minimax pay-off. The chain-store game (Figure 2(c)) and the battle-of-the-sexes game where Player 2 moves second (Figure 2(d)) have SCI. The chain-store game has SCI because, if Player 1 commits to action F and Player 2 best responds to F, then Player 1 receives a pay-off equal to four, his highest pay-off; and Player 2 receives a pay-off equal to zero, her minimax pay-off. Similarly, in the battle-of-the-sexes game where Player 2 moves second, if Player 1 plays action A and Player 2 best responds, then Player 1 receives a pay-off equal to two, his highest pay-off; and Player 2 receives a pay-off equal to one, her minimax pay-off.

Our second main contribution pinpoints why reputation effects are particularly salient in repeated games with LNCI and perfect information, whereas reputation effects are absent in certain repeated simultaneous-move games with LNCI. For example, our reputation result implies that there is a unique equilibrium pay-off profile in the repeated sequential-move battle-of-the-sexes game (Figure 1 or 2(d)). In contrast, if a simultaneous move game with LNCI, such as the battle-of-the-sexes game (Figure 3(b)), is played in each period, then a folk theorem obtains. For a more striking example, consider the repeated simultaneous-move common interest game (Figure 3), where Player 1 is potentially a Stackelberg type who always plays U. This game appears to be a strong candidate for reputation effects to arise. It is costless for Player 1 to mimic the Stackelberg type and build a reputation. Also, Player 2 unambiguously benefits if Player 1 is able to build a reputation and concentrate play on (U, L). Surprisingly, any individually rational pay-off profile can be sustained in a perfect Bayesian equilibrium if the players are arbitrarily patient (Cripps and Thomas, 1997). In contrast, we show that in the repeated sequential-move game, the players receive a pay-off equal to one, in any perfect Bayesian equilibrium.[11]

Our third main contribution is the novel method that we use to establish our reputation result. A new approach is required because the technique of Fudenberg and Levine (1989, 1992), which is commonly used to establish reputation results, is not applicable with two equally patient players. Our method hinges on having those information sets where Player 1's normal type reveals rationality be singletons (perfect information). Sequential rationality, coupled with perfect information, imposes tight bounds on Player 1's continuation pay-offs at these nodes. Moreover, for the class of games that we consider, if there is a tight bound on Player 1's continuation pay-off, then there is also a tight bound on Player 2's continuation pay-offs. These bounds preclude the possibility of Player 1 building a reputation slowly and punishing Player 2 for best responding to the Stackelberg strategy.

11. For a more detailed discussion, see Section 4.1.

2. THE MODEL

In the infinitely repeated game, a stage game Γ is played by Players 1 and 2 in periods $t \in \{0, 1, 2, \ldots\}$ and the players discount their pay-offs using a common discount factor $\delta \in [0, 1)$. *The stage game* Γ *is a two-player finite game of perfect information, i.e. all the information sets of* Γ *are singletons* (perfect information).

The set of nodes (decision nodes and terminal nodes) of the stage game Γ is denoted by $\mathcal{D}$, d is a typical element of $\mathcal{D}$, $Y \subset \mathcal{D}$ is the set of terminal nodes, and y is a typical element of Y. The pay-off function of player i is $g_i : Y \to \mathbb{R}$. The finite set of pure stage game actions for player i is A_i, and the set of mixed stage game actions is $\mathcal{A}_i$.[12] For any action profile $a = (a_1, a_2) \in A_1 \times A_2$, there is a unique terminal history $y(a) \in Y$ under the path of play induced by a. Slightly abusing notation, we let $g_i(a) = g_i(y(a))$ for any $a \in A_1 \times A_2$, and we let $g_i(\alpha)$ denote the pay-off to mixed action profile $\alpha \in \mathcal{A}_1 \times \mathcal{A}_2$.

The minimax pay-off for player i is $\hat{g}_i = \min_{\alpha_j \in \mathcal{A}_j} \max_{\alpha_i \in \mathcal{A}_i} g_i(\alpha_i, \alpha_j)$. For games that satisfy perfect information, there exists $a_1^p \in A_1$ such that $g_2(a_1^p, a_2) \leq \hat{g}_2$ for all $a_2 \in A_2$.[13] The set of feasible pay-offs F is the convex hull of the set $\{g_1(a_1, a_2), g_2(a_1, a_2) : (a_1, a_2) \in A_1 \times A_2\}$; and the set of feasible and individually rational pay-offs is $G = F \cap \{(g_1, g_2) : g_1 \geq \hat{g}_1, g_2 \geq \hat{g}_2\}$. Let $\bar{g}_1 = \max\{g_1 : (g_1, g_2) \in G\}$ and let $M = \max\{\max\{|g_1|, |g_2|\} : (g_1, g_2) \in F\}$.[14]

In the repeated game, players have perfect recall and can observe past outcomes. The set of period $t \geq 0$ public histories is denoted $Y^t \times D$ and $h = (y^0, y^1, \ldots, y^{t-1}, d)$ is a typical element. The set of period $t \geq 0$ public histories of terminal nodes is denoted $H^t \equiv Y^t$, a typical element is $h^t = (y^0, y^1, \ldots, y^{t-1})$, and we define $h^0 = \varnothing$. At the end of a period t, player i observes neither player j's stage game mixed action α_j^t in period t nor player j's pure action a_j^t. Rather, player i observes the terminal node y^t and consequently the unique sequence of moves at the decision nodes that led to the particular terminal node y^t.[15]

2.1. *Types and strategies*

Before time 0, nature selects Player 1's type ω from a countable set of types Ω according to a common knowledge prior μ. Player 2 is known with certainty to be a normal type that maximizes expected discounted utility. The set of types Ω contains a normal type for Player 1 that we denote by N. Slightly abusing notation, we denote Player 2's belief over Player 1's types after any period t public history by $\mu : \bigcup_{t=0}^{\infty} Y^t \times D \to \Delta(\Omega)$.

A behaviour strategy for player i is a function $\sigma_i : \bigcup_{t=0}^{\infty} H^t \to \mathcal{A}_i$, and Σ_i is the set of all behaviour strategies. A behaviour strategy chooses a mixed stage game action given any period t public history of terminal nodes.[16] Each type $\omega \in \Omega \setminus \{N\}$ is committed to playing a particular repeated game behaviour strategy $\sigma_1(\omega)$. A strategy profile $\sigma = (\{\sigma_1(\omega)\}_{\omega \in \Omega}, \sigma_2)$ lists the behaviour strategies of all the types of Players 1 and 2. For any period t public history

12. An action $a_i \in A_i$ is a contingent plan that specifies a move, from the set of feasible moves for player i, at any decision node d where player i is called upon to move.

13. Consider the zero sum game where Player 1's pay-off is equal to $-g_2(a_1, a_2)$. The minimax of this game is $(-\hat{g}_2, \hat{g}_2)$ by definition. Perfect information and Zermelo's lemma imply that this game has a pure strategy Nash equilibrium $(a_1^p, a_2) \in A_1 \times A_2$. Because the game is a zero sum game and the minimax value of the game is equal to $(-\hat{g}_2, \hat{g}_2)$, we have that $g_2(a_1^p, a_2) = \hat{g}_2$.

14. Note that with a slight abuse of notation, g_i denotes both the pay-off function and the pay-off level for player i.

15. See Fudenberg and Levine (1992, p. 564) for more on this particular type of imperfect monitoring inherent in extensive-form games.

16. Abusing notation, we will use σ_i to also denote mixed repeated game strategies for player i.

h^t and $\sigma_i \in \Sigma_i$, the continuation strategy induced by h^t is $\sigma_i|_{h^t}$. For $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$, the probability measure over the set of (infinite) public histories induced by (σ_1, σ_2) is $\Pr_{(\sigma_1,\sigma_2)}$.

In what follows we assume that Ω contains a certain *Stackelberg type* S. We elaborate on the Stackelberg type in Section 2.5. Also, we denote the set of *other commitment types* by $\Omega_- = \Omega \backslash \{S, N\}$. In words, Ω_- is the set of types other than the Stackelberg type and the normal type.

2.2. *The repeated game and pay-offs*

A player's repeated game pay-off is the normalized discounted sum of the stage game pay-offs. For any infinite public history $h^\infty = (y^0, y^1, \ldots)$, let $u_i(h^\infty, \delta) = (1-\delta)\sum_{k=0}^{\infty} \delta^k g_i(y^k)$; and let $u_i(h^{-t}, \delta) = (1-\delta)\sum_{k=t}^{\infty} \delta^{k-t} g_i(y^k)$ where $h^{-t} = (y^t, y^{t+1}, \ldots)$. Following a period t public history, Player 1 and Player 2's expected continuation pay-off, under strategy profile σ, are given by the following two equations, respectively:

$$U_1(\sigma, \delta|h^t) = U_1(\sigma_1(N), \sigma_2, \delta|h^t),$$

and

$$U_2(\sigma, \delta|h^t) = \sum_{\omega \in \Omega} \mu(\omega|h^t) U_2(\sigma_1(\omega), \sigma_2, \delta|h^t),$$

where $U_i(\sigma_1(\omega), \sigma_2, \delta|h^t) = \mathbb{E}_{(\sigma_1(\omega),\sigma_2)}[u_i(h^{-t}, \delta)|h^t]$ is the expectation over continuation histories h^{-t} with respect to $\Pr_{(\sigma_1(\omega)|_{h^t}, \sigma_2|_{h^t})}$. Also, $U_i(\sigma, \delta) = U_i(\sigma, \delta|h^0)$.

The repeated game of complete information (*i.e.* the repeated game without any commitment types) with discount factor equal to $\delta \in [0, 1)$ is denoted by $\Gamma^\infty(\delta)$. The repeated game of incomplete information, with the prior over the set of commitment types given by $\mu \in \Delta(\Omega)$ and the discount factor equal to $\delta \in [0, 1)$, is denoted by $\Gamma^\infty(\mu, \delta)$.

2.3. *Dynamic Stackelberg pay-off and strategy*

We define the *commitment pay-off* of Player 1's repeated game strategy σ_1 as

$$U_1^c(\sigma_1, \delta) = \min_{\sigma_2 \in BR(\sigma_1,\delta)} U_1(\sigma_1, \sigma_2, \delta),$$

where the set $BR(\sigma_1, \delta)$ denotes Player 2's best responses to σ_1 in the repeated game $\Gamma^\infty(\delta)$. Also, we define the *dynamic Stackelberg pay-off* as $U_1^s(\delta) = \sup_{\sigma_1 \in \Sigma_1} U_1^c(\sigma_1, \delta)$ and we define a *dynamic Stackelberg strategy* as any strategy, σ_1^*, that satisfies $U_1^c(\sigma_1^*, \delta) = U_1^s(\delta)$, if such a strategy exists.[17] Player 1's dynamic Stackelberg pay-off is the highest pay-off that Player 1 can secure in the repeated game through public commitment to a repeated game strategy. A dynamic Stackelberg strategy for Player 1 is a repeated game strategy such that any best response to this strategy gives Player 1 at least his dynamic Stackelberg pay-off. In other words, a dynamic Stackelberg strategy's commitment pay-off is equal to the dynamic Stackelberg pay-off.

2.4. *Class of stage games*

We assume that the stage game satisfies Assumption 1 as stated below:

17. This terminology follows Aoyagi (1996) and Evans and Thomas (1997).

Assumption 1. *The stage game Γ satisfies either of the following:*

(i) LNCI: For any $g \in G$ and $g' \in G$, if $g_1 = g'_1 = \bar{g}_1$, then $g_2 = g'_2 > \hat{g}_2$ or

(ii) SCI: There exists $a_1 \in A_1$ such that any best response to a_1 yields pay-offs $(\bar{g}_1, \hat{g}_2)$. Also, $g_2 = \hat{g}_2$ for all $(\bar{g}_1, g_2) \in G$.[18]

Both Assumption 1 item (i) and (ii) require that there is a unique pay-off profile where Player 1's pay-off is equal to $\bar{g}_1$ (*e.g.* this is true if the game Γ is a *generic* extensive form game). However, the set of games with LNCI and the set of games with SCI are mutually exclusive. Games with LNCI have a common value component: in the pay-off profile where Player 1 receives his highest pay-off and Player 2 receives a pay-off that strictly exceeds her minimax value. To see that games with LNCI have a common value component, note that in Figure 4(a), the boundary of the set of feasible pay-offs is increasing in a neighbourhood of the point $(\bar{g}_1, g_2)$. Some examples of games with LNCI are the battle-of-the-sexes game where Player 1 moves second (Figure 1), the common interest game (Figure 2(a)) and the principal–agent game (Figure 2(b)). In contrast, a game has SCI if Player 1 has an action (a Stackelberg action) such that any best response to this action yields Player 1 his highest pay-off compatible with Player 2's individual rationality and yields Player 2 her minimax pay-off.[19] Some examples of games with SCI are the chain-store game (Figure 2(c)) and the battle-of-the-sexes game where Player 1 moves first (Figure 2(d)). An example of the set of feasible pay-offs for a game with SCI is shown in Figure 4(b). Some games that do not satisfy Assumption 1 are discussed in Section 4.2.

There are two main implications of Assumption 1 that are central for the analysis that follows:

First, if Γ satisfies Assumption 1, then Player 1's dynamic Stackelberg pay-off is equal to Player 1's high pay-off that is compatible with the individual rationality of Player 2 (*i.e.* $\bar{g}_1$) and there exists a particular strategy, $\sigma_1(S)$, such that the commitment pay-off to $\sigma_1(S)$ is equal to $\bar{g}_1$, in the repeated game $\Gamma^\infty(\delta)$, for all δ that exceed a cutoff $\delta^* \in [0, 1)$. We establish this in Section 2.5 below by constructing $\sigma_1(S)$ for games that satisfy Assumption 1.[20]

Second, if Γ satisfies Assumption 1, then there are *linear bounds* on the feasible pay-offs for Player 2 that pass through the point $(\bar{g}_1, g_2)$; and hence, Player 2's pay-off are in a narrow range if Player 1's pay-off is close to $\bar{g}_1$ (see Figure 4; or for a precise statement, see inequalities (2) and (3) in Section 3.2.1). This is because Assumption 1 requires that there is a unique pay-off profile where Player 1's pay-off is equal to $\bar{g}_1$.[21]

These two main implications of Assumption 1 together establish the following (when the discount factor exceeds a cutoff $\delta^* \in [0, 1)$): *if Player 1's repeated game pay-off is close to the commitment pay-off of $\sigma_1(S)$ (which is equal to Player 1's highest pay-off compatible with Player 2's individual rationality, i.e. $\bar{g}_1$), then Player 2's feasible and individually rational repeated game pay-offs are in a narrow range determined by the linear bounds introduced in the previous paragraph.*

18. See Cripps, Dekel and Pesendorfer (2005) or Mailath and Samuelson (2006, p. 541).

19. See the product choice game (Figure 10(b) in Section 4.2) for an example where a Stackelberg action does not exist.

20. For an example in which our construction of $\sigma_1(S)$ does not work because Assumption 1 is violated, see the product choice game depicted in Figure 10(b), Section 4.2.

21. For an example that does not satisfy this requirement of Assumption 1, see the non-generic common interest game depicted in Figure 10, Section 4.2.

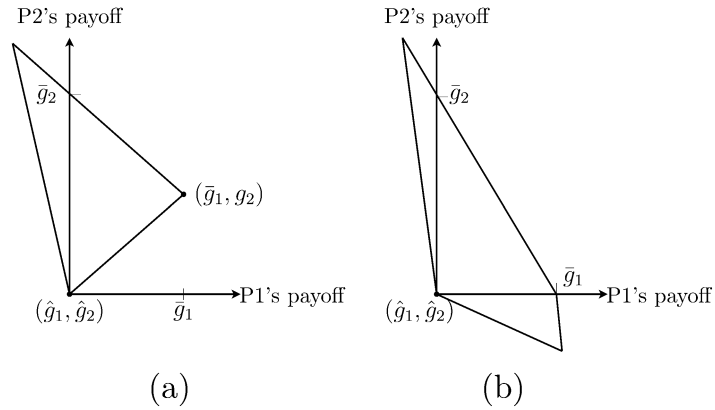


FIGURE 4

Typical set of feasible pay-offs for a game with (a) LNCI the set F is bounded above and below by the lines that go through $(\bar{g}_1, g_2)$; or (b) SCI the set F is bounded above by the downward sloping line that connects $(\bar{g}_1, \hat{g}_2)$ to $(\hat{g}_1, \bar{g}_2)$

2.5. *The Stackelberg type*

For an arbitrary stage game Γ that satisfies Assumption 1, we now construct the strategy $\sigma_1(S)$ such that $U_1^c(\sigma_1(S), \delta) = \bar{g}_1$ for all δ that exceed a cutoff δ^*.[22] We term the commitment type S who plays strategy $\sigma_1(S)$ the *Stackelberg type*.

First, some preliminary definitions: If there is an action for Player 1, $a_1 \in A_1$, and a best response for Player 2 to action a_1, $a_2 \in A_2$, such that $g_1(a_1, a_2) = \bar{g}_1$, then define $a_1^s = a_1$ and $a_2^b = a_2$.[23] Otherwise, define $(a_1^s, a_2^b) \in A_1 \times A_2$ as a particular action profile such that $g_1(a_1^s, a_2^b) = \bar{g}_1$. Assumption 1 implies that there exists an action profile $(a_1^s, a_2^b) \in A_1 \times A_2$ such that $g_1(a_1^s, a_2^b) = \bar{g}_1$.[24]

Description of $\sigma_1(S)$*:* The strategy $\sigma_1(S)$ has a profit phase and a punishment phase. In the *profit phase*, the strategy plays a_1^s, and in the *punishment phase*, the strategy plays a_1^p. The strategy begins the game in the profit phase. The strategy remains in the profit phase in period t if it was in the profit phase in period $t-1$ and if $g_1(y^{t-1}) = \bar{g}_1$. The strategy moves to the punishment phase in period t if it was in the profit phase in period $t-1$ and if $g_1(y^{t-1}) \neq \bar{g}_1$. If the strategy moves to the punishment phase in period t, then it remains in the punishment phase for $n^p - 1$ periods and then moves to the profit phase. Intuitively, $\sigma_1(S)$ punishes Player 2 by minimaxing her for the next $n^p - 1$ periods if she does not allow Player 1 to obtain a pay-off of $\bar{g}_1$. The number of punishment periods $n^p - 1$ is the *smallest integer* such that

$$g_2(a_1^s, a_2) + (n^p - 1)\hat{g}_2 < n^p g_2(a_1^s, a_2^b) \tag{1}$$

for any $a_2 \in A_2$ such that $g_1(a_1^s, a_2) < g_1(a_1^s, a_2^b) = \bar{g}_1$.

Assumption 1 implies that $n^p \geq 1$ exists. The number of punishment periods is chosen to ensure that it is a best response for a sufficiently patient Player 2 to play a_2^b in every period against $\sigma_1(S)$. More precisely, if $\sigma_2 \in BR(\sigma_1(S), \delta)$, then $U_1(\sigma_1(S), \sigma_2, \delta) = \bar{g}_1$, for all δ that

22. For games that satisfy Assumption 1, there are typically multiple dynamic Stackelberg strategies. We discuss our particular choice of $\sigma_1(S)$ and other possible dynamic Stackelberg strategies in Section 4.3.

23. If there is more than one action profile that satisfies our definition, then we pick (a_1^s, a_2^b) arbitrarily as any one of these action profiles.

24. If Γ has SCI, then a_2^b is a best response to a_1^s. If Γ has LNCI, then a_2^b is not necessarily a best response to a_1^s. For an example that satisfies Assumption 1 but where a_2^b is not a best response to a_1^s, see Figure 5.

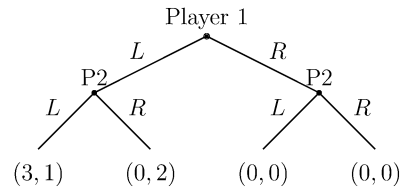


FIGURE 5

For this stage game $a_1^s = L$, $a_1^p = R$, and we pick a_2^b as the action that always chooses move L. Hence, S plays L in the profit phase, plays R in the two period punishment phase, and $n^p = 3$. Note that a_2^b is not a best response to a_1^s in this example. However, for sufficiently high δ, Player 2's best response to $\sigma_1(S)$ is to play a_2^b in each period of the repeated game. This is because playing a_2^b (instead of playing R after L) avoids the two period punishment phase

exceed a cutoff δ^*. Consequently, $\sigma_1(S)$ is a dynamic Stackelberg strategy for all δ that exceed a cutoff δ^*. For more detail, see Lemma A1 and Remark A1 in the Appendix.

If $n^p = 1$, then the strategy $\sigma_1(S)$ does not have a punishment phase, *i.e.* S is a simple type who plays the same stage game action, a_1^s, in each period of the repeated game. Moreover, Player 2's best response to $\sigma_1(S)$ entails playing a_2^b in each period for any discount factor.[25] Thus, if $n^p = 1$, then the static Stackelberg pay-off coincides with the dynamic Stackelberg pay-off for any discount factor (*e.g.* see Figure 1). If $n^p > 1$, then the dynamic Stackelberg pay-off strictly exceeds the static Stackelberg pay-off for a sufficiently high discount factor (see Figure 5).[26]

2.6. *Equilibrium and beliefs*

The analysis in the paper focuses on the perfect Bayesian equilibria (PBE) of the game of incomplete information $\Gamma^\infty(\mu, \delta)$.[27] In equilibrium, beliefs are obtained, where possible, using Bayes' rule given $\mu(\cdot|h^0) = \mu(\cdot)$ and conditioning on players' equilibrium strategies.

In what follows, *we say that Player 1 deviated from $\sigma_1(S)$ in the tth period of a period k public history h* if there exists a decision node d within period $t \leq k$ that is visited in the public history h such that the move of Player 1 in public history h at node d differs from the move that strategy $\sigma_1(S)$ would have chosen at node d. Note that if $\mu(S) > 0$, then the belief $\mu(\cdot|h)$ is well defined after any period k public history h in which Player 1 has not deviated from $\sigma_1(S)$.

3. THE REPUTATION RESULT

Our main reputation result, Theorem 1, restricts attention to stage games of perfect information that satisfy Assumption 1 and considers a repeated game $\Gamma^\infty(\mu, \delta)$ where $\mu(S) > 0$. Under these assumptions, the theorem provides a lower bound on Player 1's pay-off in any PBE. Its formal statement is given below.

Theorem 1. *Assume perfect information and Assumption 1. For any $\delta \in [0, 1)$, any $\mu \in \Delta(\Omega)$ such that $\mu(S) > 0$, and any PBE strategy profile σ of $\Gamma^\infty(\mu, \delta)$, we have*

$$U_1(\sigma, \delta) \geq \bar{g}_1 - f(\underline{z}) \max\{1 - \delta, \phi\},$$

25. Note that if $n^p = 1$ for a stage game Γ, then by rewriting inequality (1) with $n^p = 1$ we obtain $g_2(a_1^s, a_2) < g_2(a_1^s, a_2^b)$ for any $a_2 \in A_2$ such that $g_1(a_1^s, a_2) < g_1(a_1^s, a_2^b) = \bar{g}_1$, *i.e.* if $g_1(a_1^s, a_2) < \bar{g}_1$, then a_2 in not a best response to a_1^s.

26. For a definition of the static Stackelberg pay-off, see Mailath and Samuelson (2006, Chapter 15).

27. For a precise statement of PBE, see Fudenberg and Tirole (1991, Definition 8.2).

where $\underline{z} = \mu(S)$, $\phi = \mu(\Omega_-)/\mu(S)$, *and* f *is the decreasing, positive-valued function defined in equation* (A.2) *in the Appendix*.

Proof. The proof is in the Appendix. ‖

The theorem implies that as δ goes to one and $\mu(\Omega_-)$ (the probability of other commitment types) goes to zero, Player 1's pay-off converges to $\bar{g}_1$, his highest pay-off. Consequently, a normal type for Player 1 can secure a pay-off arbitrarily close to $\bar{g}_1$, his dynamic Stackelberg pay-off, in any PBE of the repeated game, for a sufficiently high discount factor and for sufficiently low probability mass on other commitment types. Player 1 can attain the bound given in the theorem by simply mimicking the Stackelberg type. Note that the bound given in the theorem is not particularly sharp, if the probability of other commitment types, $\mu(\Omega_-)$, is substantial. However, under certain assumptions, Player 1 can receive a pay-off arbitrarily close to $\bar{g}_1$, with no restrictions on the probability of other commitment types. We discuss such issues that relate to other commitment types in Section 4.4.

In order to demonstrate the implications of Theorem 1 and to make the intuition more transparently, we restate our reputation result for the example depicted in Figure 6 as Corollary 1; a detailed argument for Corollary 1 appears in Section 3.2.2. In this example, the Stackelberg type S plays U at each decision node of Player 1, and Player 1's highest stage game pay-off is equal to one. Our reputation result, for this particular example, is as follows:

Corollary 1. *Suppose that the stage game* Γ *is given by Figure* 6 *and assume that* $\mu(\Omega_-) = 0$. *For any reputation level* $\mu(S) = \underline{z} > 0$, *we have* $\lim_{\delta\to 1} U_1(\sigma(\delta), \delta) = 1$, *where* $\sigma(\delta)$ *is a PBE strategy profile for the repeated game* $\Gamma^\infty(\mu, \delta)$.

Remark 1. If the stage game Γ is given by Figure 7 instead of Figure 6, then the reputation result stated in Corollary 1 fails. We discuss this point further in Remark 2 and Section 4.1.

3.1. *The intuition for the reputation result*

We now use Figure 6 to convey the main intuition driving our reputation result. Our result shows that a sufficiently patient Player 1 can receive a pay-off approximately equal to one in any PBE by mimicking type S, *i.e.* by playing U in each period of the repeated game. Equivalently, Player 2 plays R, in only a pay-off-insignificant number of periods against an opponent who repeatedly plays U.

There are two main incentives that may induce Player 2 to play R after observing U in all previous periods. The first is a myopic incentive: she may expect Player 1 to play D with high

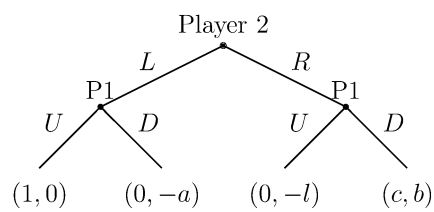


FIGURE 6

A game of perfect information with LNCI. Assume that $l \in (0, 1]$, $a \in (0, 1]$, $b \in [-1, 1/2]$ and $c \in [0, 1/2]$. If $l = a = 1$ and $b = c = 1/2$, then this is a battle-of-the-sexes game. If $l = 1$, $a = 3/4$, $b = -1$, and $c = 0$, then this is a common interest game

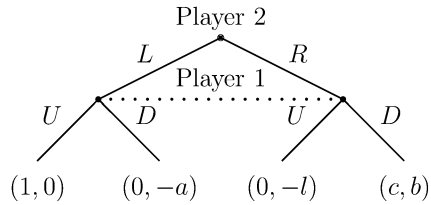


FIGURE 7

Assume that $l \in (0, 1]$, $a \in (0, 1]$, $b \in [-1, 1/2]$, and $c \in [0, 1/2]$. This simultaneous-move version of Figure 6 is a game with LNCI

probability in that period. The second is a non-myopic incentive: she may expect her continuation pay-off after R to be sufficiently more attractive than her continuation pay-off after L. We show that neither myopic nor non-myopic incentives are sufficiently strong to induce Player 2 to play R against type S for a pay-off significant number of periods. Myopic incentives are insufficient, as in Fudenberg and Levine (1989, 1992), since if Player 1 is expected to reveal rationality with high probability, then he can instead mimic type S, thereby increasing his reputation significantly and obtaining a pay-off close to one in the continuation game.

Non-myopic incentives: For Player 2 to play R in a period where Player 1 plays D with small probability, she must expect a punishment for playing L (or a reward for playing R) in the continuation game. Type S always plays U; hence, any punishment (or reward) for Player 2 must occur after Player 1 reveals rationality by playing D. Because Player 1 moves after observing Player 2's move (perfect information), he can continue to mimic type S instead of punishing Player 2 after observing L (or rewarding her after R). Hence, his pay-off while punishing (or rewarding) Player 2 cannot differ significantly from his pay-off from mimicking type S. In other words, punishing (or rewarding) Player 2 cannot be costly for Player 1. For the class of games that we consider, the commitment pay-off of type S is equal to the highest pay-off of Player 1. Moreover, for this class of games, if Player 1's pay-off is close to his highest pay-off, then Player 2's pay-offs are in a narrow range (also see Figure 4(a)). Therefore, if punishments (or rewards) are not costly for Player 1, then Player 2's feasible continuation pay-offs lie in a narrow range. Thus, the scope for non-myopic incentives is also limited.

3.2. *The argument for the reputation result*

3.2.1. Preliminaries. Recall that $(a_1^s, a_2^b) \in A_1 \times A_2$ is an action profile such that $g_1(a_1^s, a_2^b)$ is equal to Player 1's highest stage game pay-off compatible with individual rationality. For Figure 6, the stage game action a_1^s plays U after either L or R; and a_2^b is the best response to a_1^s, *i.e.* $a_2^b = L$. Also, for this game $n^p = 1$, *i.e.* the static and dynamic Stackelberg pay-offs coincide and are equal to one, for any discount factor.

If Γ satisfies Assumption 1(i), then there exists a finite constant $\rho \geq 0$ such that

$$|g_2 - g_2(a_1^s, a_2^b)| \leq \rho |\bar{g}_1 - g_1| \quad \text{for any } (g_1, g_2) \in F. \tag{2}$$

For example in Figure 6, any feasible pay-off profile (g_1, g_2) satisfies inequality (2) for $\rho = 1$. Also, see Figure 4(a) for a depiction of inequality (2). The set of feasible pay-offs in the repeated game is equal to the set of feasible stage game pay-offs. Therefore, if Γ satisfies Assumption 1 (i), then inequality (2) implies that

$$|U_2(\sigma_1, \sigma_2, \delta) - g_2(a_1^s, a_2^b)| \leq \rho |\bar{g}_1 - U_1(\sigma_1, \sigma_2, \delta)| \quad \text{for any pair } (\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2.$$

If Γ satisfies Assumption 1 (ii), then there exists a finite constant $\rho \geq 0$ such that

$$g_2 - g_2(a_1^s, a_2^b) \leq \rho(\bar{g}_1 - g_1) \quad \text{for any } (g_1, g_2) \in F. \tag{3}$$

Also, see Figure 4(a) for a depiction of inequality (3). If Γ satisfies Assumption 1 (ii), then inequality (3) implies that

$$U_2(\sigma_1, \sigma_2, \delta) - g_2(a_1^s, a_2^b) \leq \rho(\bar{g}_1 - U_1(\sigma_1, \sigma_2, \delta)) \quad \text{for any pair } (\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2.$$

We now introduce the **resistance function**, $R(\mu, \delta)$, which is central to the analysis that follows. As a preliminary, we define the resistance of the strategy σ_2 for Player 2 as follows:

$$r(\sigma_2, \delta) = \bar{g}_1 - U_1(\sigma_1(S), \sigma_2, \delta).$$

For the example in Figure 6, the resistance of strategy σ_2 is equal to the expected discounted number of periods in which (R, U) is played under the strategy profile $(\sigma_1(S), \sigma_2)$. That is, the resistance of strategy σ_2 is equal to the expected number of times a non-best response is played by strategy σ_2 against the Stackleberg type. Note that if Player 2 uses strategy σ_2 and her opponent uses strategy $\sigma_1(S)$, then Player 2's pay-off, $U_2(\sigma_1(S), \sigma_2, \delta)$, is equal to $-lr(\sigma_2, \delta)$. This is because either (R, U) or (L, U) is played in each period; and $g_2(R, U) = -l$ and $g_2(L, U) = 0$.

The resistance function, $R(\mu, \delta)$, provides an upper bound on how much Player 2 can resist (or hurt) type S in *any PBE* of $\Gamma^\infty(\mu, \delta)$. It is defined as follows:

Definition 1 (**Resistance Function**). *For any measure $\mu \in \Delta(\Omega)$ and $\delta \in [0, 1)$, let*

$$R(\mu, \delta) = \sup\{r(\sigma_2, \delta) : \sigma_2 \textit{ is part of a PBE profile } \sigma \textit{ of } \Gamma^\infty(\mu, \delta)\}.$$

3.2.2. The argument for Corollary 1. In this subsection, we prove the reputation result given in Corollary 1. At the end of the section, we discuss the main argument for Theorem 1 that is given in the Appendix. In what follows, because $\mu(\Omega_-) = 0$, we use $z \in [0, 1]$ to represent the measure μ. One should understand this to mean $\mu(S) = z$ and $\mu(N) = 1 - z$.

In this section, we work under the hypothesis that the resistance function $R(z, \delta)$ *is a non-increasing function of z for each $\delta \in [0, 1)$*. We do this for expositional convenience only, as it allows us to convey the main argument without the more technical details.[28]

At the start of any period t, if Player 1's reputation level is at least $z > 0$, then Player 1 can guarantee a continuation pay-off of at least $1 - R(z, \delta)$ by playing according to the Stackelberg strategy $\sigma_1(S)$. This follows from the definition of R, sequential rationality, and our assumption that R is non-increasing. We will argue that $\lim_{\delta \to 1} R(\underline{z}, \delta) = 0$ for any $\underline{z} > 0$.

Consider a PBE σ of the repeated game $\Gamma^\infty(z, \delta)$. Suppose that the players are at a history in which Player 1 has played U in each period before t, and Player 2 has played $a_2 \in \{L, R\}$ in period t. Suppose further that Player 1 plays D with positive probability at this decision node, *i.e.* Player 1 reveals that he is not the Stackelberg type, with positive probability. Also, let Player 1's reputation level be $z' > 0$ at the start of period $t + 1$, if he plays U instead of D. In the next lemma, we bound the continuation pay-off of Player 2 by a linear function of $R(z', \delta)$ at any such decision node. The argument for the lemma is as follows: If Player 1 is playing D with positive

28. In the Appendix, we instead work with the maximal resistance function $\bar{R}(z, \delta) = \sup\{R(z', \delta) : z' \geq z\}$, which is non-increasing by definition.

probability, then the pay-off from playing D must be at least as large as the pay-off from playing U. However, if Player 1 plays U, he gets at worst zero for the period, ensures that his reputation is z' at the start of the subsequent period, and thus guarantees $1 - R(z', \delta)$ at the start of period $t+1$. Given this lower bound on Player 1's continuation pay-off, the linear bound on Player 2's continuation pay-off follows from inequality (2).

Lemma 1. *Suppose $z > 0$. Pick any PBE σ of $\Gamma^{\infty}(z, \delta)$ and any period t public history of terminal nodes h^t where Player 1 has played U in each period; and suppose Player 1 plays D in period t given history (h^t, a_2) with positive probability, where $a_2 \in \{L, R\}$. Let $z' = \mu(S|h^t, a_2, U)$; then, we have*

$$|U_2(\sigma_1(N), \sigma_2, \delta|h^t, a_2, D)| \leq R(z', \delta) + (1-\delta)/\delta.$$

Proof. If Player 1 plays U in period t, then his reputation level is $z' = \mu(S|h^t, a_2, U)$ and he can guarantee a continuation pay-off equal to $1 - R(z', \delta)$, by using $\sigma_1(S)$. Also, Player 1 can get at worst zero in period t by playing U. Consequently, his pay-off from playing U is at least $\delta U_1(\sigma, \delta|h^t, a_2, U) \geq \delta(1 - R(z', \delta))$. If Player 1 plays D instead, then he can get at most c for the current period and $\delta U_1(\sigma, \delta|h^t, a_2, D)$ as his continuation pay-off. Because Player 1 is willing to play D instead of U, we have $(1-\delta)c + \delta U_1(\sigma, \delta|h^t, a_2, D) \geq \delta U_1(\sigma, \delta|h^t, a_2, U)$. Hence, $U_1(\sigma, \delta|h^t, a_2, D) \geq 1 - R(z', \delta) - (1-\delta)c/\delta \geq 1 - R(z', \delta) - (1-\delta)/\delta$. The bound on Player 2's pay-off follows from inequality (2) because the pay-off profile $(U_1(\sigma, \delta|h^t, a_2, D), U_2(\sigma_1(N), \sigma_2, \delta|h^t, a_2, D))$ is an element of F and because the constant ρ is at most one for this particular game. Also see Figure 8. ‖

Remark 2. Lemma 1 puts a bound on $U_2(\sigma_1(N), \sigma_2, \delta|h^t, a_2, D)$ not only for Player 2's equilibrium choice of a_2 but also for any $a_2 \in \{L, R\}$. Instead of the game of perfect information in Figure 6, suppose that the stage game Γ is given by the simultaneous-move game in Figure 7. Further suppose, in a given PBE σ after history h^t, Player 2 plays R with probability one and Player 1 plays D with positive probability. Then, as in Lemma 1, Player 1's *ex ante* incentive constraint implies that $U_1(\sigma, \delta|h^t, R, D) \geq 1 - R(z', \delta) - (1-\delta)/\delta$. However, in

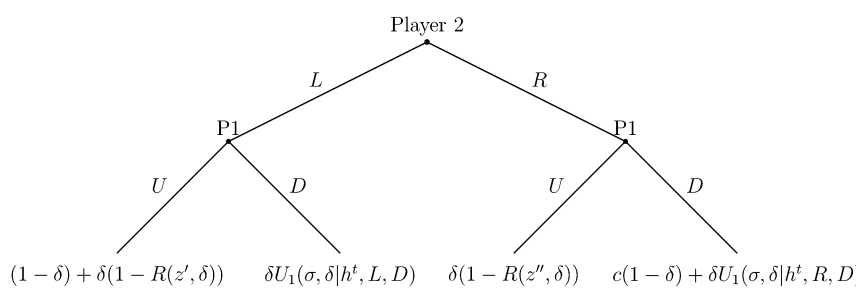


FIGURE 8

This figure depicts the pay-off Player 1 can guarantee by playing U and his pay-off if he plays D instead. In the figure, $z' = \mu(S|h^t, L, U)$ and $z'' = \mu(S|h^t, R, U)$. If Player 1 is to play D with positive probability after R, then $(1-\delta)c + \delta U_1(\sigma, \delta|h^t, R, D) \geq \delta(1 - R(z'', \delta))$. Consequently, $U_1(\sigma, \delta|h^t, R, D) \geq 1 - R(z'', \delta) - (1-\delta)c/\delta$, inequality (2), and $\rho \leq 1$ together imply that $|U_2(\sigma, \delta|h^t, R, D)| \leq R(z'', \delta) + (1-\delta)c/\delta$. Similarly, if Player 1 is to play D after L, then $|U_2(\sigma, \delta|h^t, L, D)| \leq R(z', \delta) - (1-\delta)/\delta$

contrast to Lemma 1, it is no longer possible to assert that Player 1's *ex ante* incentives require $U_1(\sigma, \delta|h^t, L, D) \geq 1 - R(z', \delta) - (1-\delta)/\delta$; and as a consequence, it is not possible to assert that $|U_2(\sigma_1(N), \sigma_2, \delta|h^t, L, D)| \leq \rho(R(z', \delta) + (1-\delta)/\delta)$. This is because Player 1 chooses to play D before seeing Player 2's move (*i.e.* perfect information is violated) and expects Player 2 to play L with probability zero when making his choice; therefore, Player 1's continuation pay-off after (h^t, L, D) does not affect his *ex ante* incentives. We discuss this point further in Section 4.1.

We now use Lemma 1 to *sketch the argument for Corollary 1*. Suppose that Player 1's reputation level is z. Consider a PBE $\sigma = (\sigma_1(N), \sigma_1(S), \sigma_2)$ where Player 2 resists the Stackelberg type by approximately $R(z, \delta)$. In this PBE, Player 2 loses approximately $lR(z, \delta)$ in the event that Player 1 is the Stackelberg type. We compare Player 2's pay-off in this PBE with her pay-off if she uses an *alternative strategy* that plays L until Player 1 plays D for the first time and then reverts back to the equilibrium strategy σ_2. If Player 2 uses the alternative strategy, then she avoids losing $lR(z, \delta)$ in the event that Player 1 is the Stackelberg type. We then use the fact that the PBE strategy σ_2 must give Player 2 a pay-off that is at least as great as the pay-off from using the alternative strategy. This establishes a bound on $R(z, \delta)$ for any z sufficiently close to 1.

Upper bound on Player 2's equilibrium pay-off. Suppose that Player 1 plays D for the first time in some period t. In each period up to period t, Player 2 receives at most zero; in period t, she receives at most $1-\delta$; and after period t, she receives at most $R(z, \delta) + (1-\delta)/\delta$ as a continuation pay-off, by Lemma 1 and because R is non-increasing. Consequently, Player 2 gets at most $\delta^t(1-\delta) + \delta^{t+1}(R(z, \delta) + (1-\delta)/\delta) \leq R(z, \delta) + 2(1-\delta)$, if Player 1 plays D for the first time in period t. Alternatively, if Player 1 plays U in each period, then Player 2 receives at most $-lR(z, \delta)$. Player 1 will play U in every period with probability at least z because type S always plays U. So, Player 1 will play D in some period t, with probability at most $1-z$. Thus, Player 2's pay-off in the PBE σ is at most $(1-z)(R(z, \delta) + 2(1-\delta)) - zlR(z, \delta)$. This line of reasoning is formalized by the upper bound that we establish in Lemma 2 further below.

Lower bound on Player 2's equilibrium pay-off. Suppose that Player 2 uses the alternative strategy and Player 1 plays D for the first time in some period k. Player 2 receives at least $-R(z, \delta) - (1-\delta)/\delta$ as a continuation pay-off after period k, by Lemma 1 and because R is non-increasing. Also, she receives zero in each period up to period k because she plays L and Player 1 plays U. In period k, she receives $-a(1-\delta) \geq -(1-\delta)$ because she plays L and Player 1 plays D and because $a \in (0, 1]$. Alternatively, if Player 1 plays U in every period, then Player 2 receives zero. Player 1 will play D in some period k, with probability at most $1-z$. Consequently, if Player 2 uses the alternative strategy, then her pay-off is at least $-(1-z)(\delta^k(1-\delta) + \delta^{k+1}(R(z, \delta) + (1-\delta)/\delta)) \geq -(1-z)(R(z, \delta) + 2(1-\delta))$. This line of reasoning is formalized by the lower bound that we establish in Lemma 3 further below.

Bounding resistance. The pay-off that Player 2 gets from the equilibrium strategy σ_2 must be at least as great as the pay-off she receives from the alternative strategy. So, $-(1-z)(R(z, \delta) + 2(1-\delta)) \leq (1-z)(R(z, \delta) + 2(1-\delta)) - zlR(z, \delta)$. Rearranging, $R(z, \delta) \leq 4(1-z)(1-\delta)/(lz - 2(1-z)) \leq 4(1-\delta)/(lz - 2(1-z))$. Thus, for z sufficiently close to one, *i.e.* if $1-z \leq \underline{q} \equiv l\underline{z}/4$, then $R(z, \delta) \leq C(1-\delta)$ where $C = 16/\underline{z}l$.[29] Therefore, the resistance at reputation level z is very close to zero if δ is close to one.

More generally, the argument for Corollary 1 shows that for any two reputation levels $z'' > z' \geq \underline{z}$ such that $z'/z'' \geq 1 - \underline{q}$, the resistance function satisfies the following *functional*

29. To be precise, if $z \geq \underline{z}$ and $1-z \leq l\underline{z}/4$, then $lz - 2(1-z) \geq l\underline{z}/2$. Hence, $R(z, \delta) \leq 4(1-\delta)/l\underline{z}/2 = 8(1-\delta)/l\underline{z} \leq 16(1-\delta)/l\underline{z}$.

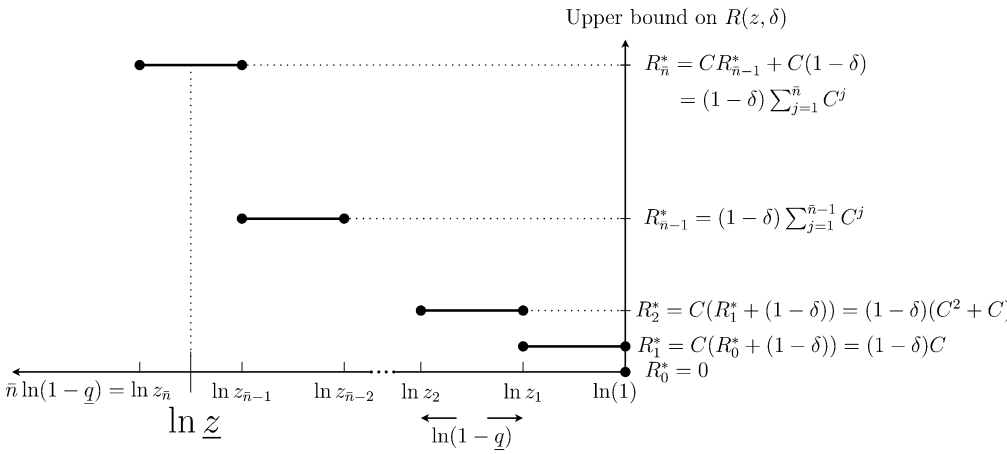


FIGURE 9

Resistance's upper bound as a function of $\ln(z)$. This upper bound is implied by the functional inequality $R(z',\delta) \leq CR(z'',\delta) + C(1-\delta)$ that holds for any $z'' > z' \geq \underline{z}$ such that $z'/z'' \geq 1-\underline{q}$. The reputation levels z_n, that are shown on the x-axis, are such that $\ln z_n - \ln z_{n+1} = \ln(1-\underline{q})$ for each n

inequality: $R(z',\delta) \leq CR(z'',\delta) + C(1-\delta)$. That is, decreasing Player 1's reputation level by a factor of $1-\underline{q}$ increases resistance by at most a factor of C plus an additive term equal to $C(1-\delta)$. Observe that we can recover the inequality that we established in the previous paragraph by substituting $z' = 1-\underline{q}$ and $z'' = 1$ into the functional inequality and by using the fact that $R(1,\delta) = 0$. Figure 9 depicts the upper bound on resistance implied by this functional inequality. Note that for any reputation level $z \geq \underline{z}$, the upper bound on resistance depicted in Figure 9, converges to zero, as δ converges to one.

We now proceed with the formal proof of Corollary 1 by establishing an upper and a lower bound (Lemmata 2 and 3) for Player 2's PBE pay-offs. Definition 2 below introduces a stopping time which we use in constructing the upper and the lower bound.

Definition 2 (Stopping Time). *For an integer k, let $E_{[0,k]}$ denote the event (set of infinite public histories) where Player 1 plays D for the first time in period t for some $t \in \{0,\ldots,k\}$. For any strategy profile $\sigma = (\sigma_1, \sigma_1(S), \sigma_2)$, and for reputation levels $z > 0$ and $z' > z$, let*

$$T(\sigma, z, z') = \min\{k \in \{0,1,2,\ldots\} : z/(1-\pi(k)) \geq z'\},$$

where $\pi(k) = (1-z)\Pr_{(\sigma_1,\sigma_2)}[E_{[0,k]}]$; and let $T(\sigma,z,z') = \infty$ if the set is empty. Note that $\Pr_{(\sigma_1,\sigma_2)}[E_{[0,k]}]$ is the probability that Player 1 plays D for the first time in period t for some $t \in \{0,\ldots,k\}$, if Player 1 is using strategy σ_1 and Player 2 is using strategy σ_2.

Intuitively, the stopping time $T(\sigma,z,z')$ gives the first period in which Player 1's reputation level exceeds z', if his initial reputation level is z and if the players use strategy profile σ. The specific implications of Definition 2, that we use in Lemmata 2 and 3, are summarized in the following remark:

Remark 3. Suppose that Player 1's initial reputation level is z and suppose that $z' > z$. Let $\sigma^* = (\sigma_1(N), \sigma_1(S), \sigma_2^*)$ be any strategy profile where σ_2^* is a pure strategy. Let $T = T(\sigma^*, z, z')$. By definition, the total probability that Player 1 plays D for the first time in any period $t \leq T-1$

(*i.e.* $\pi(T-1)$ in the terminology used in Definition 2) is at most $1-z/z'$. Also, because both $\sigma_1(S)$ and σ_2^* are pure strategies, we have the following:

(i) There is a unique public history of terminal nodes h^{T+1} consistent with strategies $\sigma_1(S)$ and σ_2^*.

(ii) If h^{T+1} is the unique history consistent with $\sigma_1(S)$ and σ_2^* (*i.e.* Player 1 has always played U in all periods up to and including period T), then Bayes' rule implies that $\mu(S|h^{T+1}) \geq z'$.

Lemma 2 (Upper Bound). *Suppose* $0 \leq z < z' \leq 1$. *Let* $\sigma = (\sigma_1(N), \sigma_1(S), \sigma_2)$ *denote a PBE of* $\Gamma^\infty(z,\delta)$ *where Player 2's resistance is at least* $R(z,\delta)-\epsilon$ *and* $\epsilon > 0$. *Then,*

$$U_2(\sigma,\delta) \leq q(R(z,\delta)+2(1-\delta)) + R(z',\delta) + 2(1-\delta) - zl(R(z,\delta)-\epsilon), \tag{4}$$

where $q = 1-z/z'$.

Proof. Let σ_2^* denote a pure strategy in the support of σ_2 such that the resistance of σ_2^* is at least $R(z,\delta)-\epsilon$. Since the resistance of σ_2 is at least $R(z,\delta)-\epsilon$, there must be a pure strategy in the support of σ_2 that has resistance of at least $R(z,\delta)-\epsilon$. Let profile $\sigma^* = (\sigma_1(N), \sigma_1(S), \sigma_2^*)$ and let $T = T(\sigma^*, z, z')$. As we argued in Remark 3, Player 1's reputation exceeds z' at the end of period T if Player 1 plays U and if Player 2 plays according to σ_2^* in all periods up to and including T.

We bound Player 2's pay-offs from σ_2^* in the following three events: (i) the event where Player 1 plays D for the first time in some period $t < T$; the probability of this event is at most $q = 1-z/z'$ by Remark 3; (ii) the event where Player 1 plays D for the first time in some period $t \geq T$; the probability of this event is at most 1; (iii) the event where Player 1 never plays D; the probability of this event is at least z because S never plays D. These three events are exhaustive.

In a period where Player 1 plays U, Player 2 receives at most zero. Consequently, Player 2's total pay-off in all the periods until Player 1 plays D for the first time is at most zero.

If event (i) occurs and Player 1 plays D for the first time in some period t, then Player 2 receives zero until period t, receives at most $(1-\delta)$ in period t,[30] and knows with certainty that she faces the normal type N. Hence, she receives a continuation pay-off of at most $R(z,\delta) + (1-\delta)/\delta$ by Lemma 1 and because R is non-increasing. So if event (i) occurs, then Player 2's pay-off is at most $R(z,\delta)+2(1-\delta)$ because

$$R(z,\delta)+2(1-\delta) \geq \delta^t(1-\delta) + \delta^{t+1}(R(z,\delta)+(1-\delta)/\delta) \quad \text{for any period } t.$$

If event (ii) occurs and Player 1 plays D for the first time in some period t, then Player 2 receives zero until period t, receives at most $(1-\delta)$ in period t, and receives a continuation pay-off of at most $R(z',\delta)+(1-\delta)/\delta$, by Lemma 1 and because R is non-increasing. So if event (ii) occurs, then Player 2's pay-off is at most $R(z',\delta)+2(1-\delta)$ because

$$R(z',\delta)+2(1-\delta) \geq \delta^t(1-\delta) + \delta^{t+1}(R(z',\delta)+(1-\delta)/\delta) \quad \text{for any period } t.$$

If event (iii) occurs, then Player 1 plays U in each period; Player 2's pay-off in this event is at most $-l(R(z,\delta)-\epsilon)$. This is because the resistance of σ_2^* is at least $R(z,\delta)-\epsilon$.

Putting the bounds on Player 2's pay-offs in the three events together, we obtain

$$U_2(\sigma^*,\delta) \leq q(R(z,\delta)+2(1-\delta)) + R(z',\delta) + 2(1-\delta) - zl(R(z,\delta)-\epsilon).$$

30. Player 2's highest stage game pay-off is one in this game.

Recall that σ_2^* is in the support of PBE strategy σ_2. Consequently, we have $U_2(\sigma^*, \delta) = U_2(\sigma, \delta)$, which implies the following:

$$U_2(\sigma, \delta) \leq q(R(z, \delta) + 2(1-\delta)) + R(z', \delta) + 2(1-\delta) - zl(R(z, \delta) - \epsilon).$$

‖

Lemma 3 (Lower Bound). *Suppose* $0 \leq z < z' \leq 1$. *In any PBE* σ *of* $\Gamma^\infty(z, \delta)$, *we have*

$$U_2(\sigma, \delta) \geq -q(R(z, \delta) + 2(1-\delta)) - R(z', \delta) - 2(1-\delta), \tag{5}$$

where $q = 1 - z/z'$.

Proof. Pick any PBE σ of $\Gamma^\infty(z, \delta)$. Let σ_2^* denote a pure strategy that moves according to a_2^b after any period k public history h^k that is consistent with $\sigma_1(S)$; and that coincides with a pure strategy in the support of the PBE strategy σ_2 if h^k is not consistent with $\sigma_1(S)$. Let strategy profile $\sigma^* = (\sigma_1(N), \sigma_1(S), \sigma_2^*)$ and let $T = T(\sigma^*, z, z')$. As we argued in Remark 3, Player 1's reputation exceeds z' at the end of period T if Player 1 plays U and if Player 2 plays according to σ_2^*, in all periods up to and including T.

Once again we bound Player 2's pay-off from strategy σ_2^* in the following three events: (i) The event where Player 1 plays D for the first time in some period $t < T$; the probability of this event is at most q, by Remark 3; (ii) the event where Player 1 plays D for the first time in some period $t \geq T$; the probability of this event is at most 1; (iii) the event that Player 1 never plays D.

Player 2's pay-off until Player 1 plays D for the first time is at most zero. If event (i) occurs and Player 1 plays D for the first time in some period t, then Player 2 receives zero until period t, receives at worst $-a(1-\delta) \geq -(1-\delta)$ in period t, and receives a continuation pay-off of at worst $-R(z, \delta) - (1-\delta)/\delta$, by Lemma 1 and because R is non-increasing. Consequently, Player 2's pay-off is at least $-R(z, \delta) - 2(1-\delta)$ because

$$-\delta^t(1-\delta) - \delta^{t+1}(R(z, \delta) + (1-\delta)/\delta) \geq -R(z, \delta) - 2(1-\delta).$$

If event (ii) occurs and Player 1 plays D for the first time in some period t, then Player 2 receives zero until period t, receives at worst $-(1-\delta)$ in period t, and receives a continuation pay-off of at worst $-R(z', \delta) - (1-\delta)/\delta$ by Lemma 1 and because R is non-increasing. Consequently, Player 2's pay-off is at least $-R(z', \delta) - 2(1-\delta)$ because

$$-\delta^t(1-\delta) - \delta^{t+1}(R(z', \delta) + (1-\delta)/\delta) \geq -R(z', \delta) - 2(1-\delta).$$

If event (iii) occurs, then Player 1 never plays D and consequently Player 2 receives zero. Putting the bounds on Player 2's pay-offs in the three events together implies that

$$U_2(\sigma^*, \delta) \geq -q(R(z, \delta) + 2(1-\delta)) - R(z', \delta) - 2(1-\delta).$$

Because σ_2 is Player 2's PBE strategy, we have $U_2(\sigma, \delta) \geq U_2(\sigma^*, \delta)$. Consequently,

$$U_2(\sigma, \delta) \geq -q(R(z, \delta) + 2(1-\delta)) - R(z', \delta) - 2(1-\delta).$$

‖

Below we use the fact that the upper bound provided in Lemma 2 must exceed the lower bound given in Lemma 3 to obtain a functional inequality that relates maximal resistance at any two reputation levels. We then use this functional inequality to complete our proof.

Lemma 4 (**Functional Inequality**). *For any $z \in [\underline{z}, 1]$ and $z < z' \leq 1$, we have*

$$R(z,\delta)(\underline{z}l - 2q) \leq 2R(z',\delta) + 8(1-\delta), \tag{6}$$

where $q = 1 - z/z'$.

Proof. For any $\epsilon > 0$, there exists a PBE σ of $\Gamma^{\infty}(z,\delta)$ where Player 2's resistance is at least $R(z,\delta) - \epsilon$ by the definition of the resistance function. By Lemma 2, inequality (4) holds for any $\epsilon > 0$ and any PBE σ where Player 2's resistance is at least $R(z,\delta) - \epsilon$. Also, the upper bound given by inequality (4) must exceed the lower bound given by inequality (5) for any PBE σ. Combining (4) and (5), taking the limit as ϵ goes to zero, and substituting $\underline{z}$ for z together imply that $R(z,\delta)(\underline{z}l - 2q) \leq 4R(z',\delta) + 4(1+q)(1-\delta)$. Using $q \leq 1$ then delivers inequality (6). ‖

Proof of Corollary 1 under the hypothesis that R is non-increasing. Let $\underline{q} = \underline{z}l/4$ and let $\bar{n}$ be the smallest integer such that $(1-\underline{q})^{\bar{n}} \leq \underline{z}$. We will show that $R(\underline{z},\delta) \leq (1-\delta)\sum_{j=1}^{\bar{n}} C^j$ where $C = 16/\underline{z}l$ and hence $0 \leq \lim_{\delta\to 1} R(\underline{z},\delta) \leq \lim_{\delta\to 1}(1-\delta)\sum_{j=1}^{\bar{n}} C^j = 0$.

If $z, z' \in [\underline{z}, 1]$ and $z \in [z'(1-\underline{q}), z']$, then $1 - z/z' \leq \underline{q}$. Hence, substituting $\underline{q}$ for $q = 1 - z/z'$ in inequality (6) delivers the following:

$$R(z,\delta)(\underline{z}l - 2\underline{q}) \leq 2R(z',\delta) + 8(1-\delta).$$

Substituting $\underline{z}l/4$ for $\underline{q}$ in the previous inequality and rearranging we obtain the following:

$$R(z,\delta) \leq \frac{4}{\underline{z}l}(R(z',\delta) + 4(1-\delta)).$$

Substituting C for $16/\underline{z}l$ in the previous inequality and using the fact that $R(z',\delta) \geq 0$, we obtain the following:

$$R(z,\delta) \leq CR(z',\delta) + C(1-\delta). \tag{7}$$

If $z \geq \underline{z}$ and $z \in [1-\underline{q}, 1]$, then substituting $z' = 1$ into inequality (7), we obtain the following:

$$R(z,\delta) \leq CR(1,\delta) + C(1-\delta),$$

for $z, z' \in [\underline{z}, 1]$ such that $z \in [z'(1-\underline{q}), z']$. Note that $R(1,\delta) = 0$. Consequently, if $z \geq \underline{z}$ and $z \in [1-\underline{q}, 1]$, then $R(z,\delta) \leq C(1-\delta)$.

More generally, we will show that if $z \geq \underline{z}$ and if $z \in [(1-\underline{q})^n, (1-\underline{q})^{n-1}]$, then $R(z,\delta) \leq (1-\delta)\sum_{j=1}^{n} C^j$ by using induction on n. We make the inductive hypothesis that if $z \geq \underline{z}$ and if $z \in [(1-\underline{q})^{k-1}, (1-\underline{q})^{k-2}]$, then we have $R(z,\delta) \leq (1-\delta)\sum_{j=1}^{k-1} C^j$.

If $z \geq \underline{z}$ and $z \in [(1-\underline{q})^k, (1-\underline{q})^{k-1}]$, then substituting $(1-\underline{q})^{k-1}$ for z' in inequality (7) gives us the following:

$$R(z,\delta) \leq CR((1-\underline{q})^{k-1},\delta) + C(1-\delta). \tag{8}$$

However, inequality (8) and the inductive hypothesis together show that if $z \geq \underline{z}$ and $z \in [(1-\underline{q})^k, (1-\underline{q})^{k-1}]$, then $R(z,\delta) \leq (1-\delta)\sum_{j=1}^{k} C^j$, completing the induction.

The definition of $\bar{n}$ implies that $\underline{z} \in [(1-\underline{q})^{\bar{n}}, (1-\underline{q})^{\bar{n}-1}]$, and consequently, $R(\underline{z},\delta) \leq (1-\delta)\sum_{j=1}^{\bar{n}} C^j$. See Figure 9 for a depiction of this argument. ‖

3.2.3. Description of the proof of Theorem 1. Our discussion up to this point established a reputation result for the game depicted in Figure 6. Here, we describe the additional arguments we use to prove Theorem 1. In particular, we sketch the steps involved in allowing for the Stackelberg type that uses punishments (*i.e.* $n^p > 1$) and allowing for other commitment types (*i.e.* $\mu(\Omega_-) > 0$).

In order to accommodate the Stackelberg type who punishes player 2, *i.e.* the case where $n^p > 1$, we prove Lemmata A1 and A2. Lemma A1 shows that Player 2 faces an average per-period cost, $l > 0$, for not best responding to the Stackelberg type. Lemma A2 is an analog of Lemma 1 that accounts for punishment phases. This lemma is needed because at any node where Player 1 deviates from $\sigma_1(S)$ under equilibrium play, if he instead plays according to $\sigma_1(S)$ in order to maintain his reputation, then he may have to carry out an $n_p - 1$ period punishment phase.

Allowing for $\mu(\Omega_-) > 0$ requires accounting for the event where Player 2 faces another commitment type in the lower and upper bound calculations. In particular, we show that the effect of the other commitment types is at most $\pm M\mu(\Omega_-)$ on the lower bound and the upper bound. This is because Player 1 is another commitment type with probability $\mu(\Omega_-)$ and because Player 2 can gain or lose at most M against any type. Consequently, if $\mu(\Omega_-)$ is small, then the effect of other commitment types on the functional inequality is also small.

4. DISCUSSION

4.1. *Without perfect information, the reputation result can fail*

For example, without perfect information, a folk theorem applies to the simultaneous-move common interest game in Figure 3 (Cripps and Thomas, 1997), which has LNCI. For a reputation result in repeated games with SCI, the perfect information assumption is not required (Cripps, Dekel and Pesendorfer, 2005 or Section 4.6 in this paper).

Corollary 1 provides a reputation result for the repeated sequential common interest game.[31] Lemma 1 is central for establishing Corollary 1 and the perfect information assumption is required for Lemma 1. In order to flesh out the intuition of why perfect information is necessary, we construct a PBE for the repeated simultaneous-move common interest game given in Figure 3 (where we take $x = 0$), where there is no analog of Lemma 1. In this PBE, the players' pay-offs are low if z is close to zero and δ is close to one.[32] That is, the failure of Lemma 1 also leads to the failure of the reputation result.

Suppose Player 2 plays R and Player 1 uses a mixed strategy that plays D with small probability for the first K periods. After the first K periods, (L, U) is played forever. In this construction, $U_1(\sigma) = U_2(\sigma) = \delta^K$. Also, the continuation pay-off for the players, after (R, D) or (R, U), is equal to δ^{K-t} in any period $t \in \{0, \ldots, K-1\}$. To ensure that Player 2 has an incentive to play R, she is punished in the event that she plays L and Player 1 plays D (thus revealing rationality). Punishment entails a continuation pay-off for Player 2 that is close to zero.[33] Player 1 is willing to mix between U and D in the first K periods since Player 2 only plays R on the equilibrium path.

In this construction, Player 2 is deterred from playing L, even if Player 1 reveals rationality with a small probability in each period because her continuation pay-off is close to zero at (L, D). However, if the probability that Player 1 reveals rationality is small in each period, then

31. This is because Figure 6 is a normalized sequential common interest game if $a = 1$ and $b = -1$.

32. This construction follows Cripps and Thomas (1997).

33. After (L, D) or (R, D), the continuation game is a repeated game of complete information and any pay-off in $[0, 1]$ can be supported in equilibrium.

it takes many periods for Player 1 to build a reputation and K can be chosen large to ensure low pay-offs for both players.

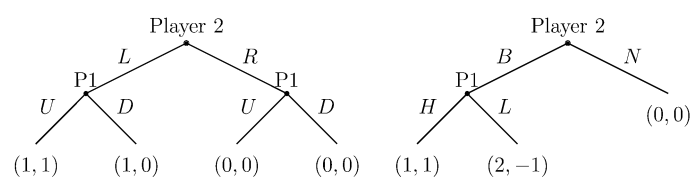
This argument hinges on choosing low continuation pay-offs for Player 2 after terminal node (L, D) during the first K periods. This does not conflict with Player 1's incentive to play D instead of U, even if low continuation pay-offs for Player 2 also implies low continuation pay-offs for Player 1, after node (L, D). This is because, in the first K periods, when Player 1 makes his move, he expects Player 2 to play L with probability zero and the terminal node (L, D) is reached with probability zero. Thus, *pay-offs at node* (L, D) *have no effect on Player 1's ex ante incentive to play D and consequently Player 1's incentive constraint puts no restrictions on Player 2's continuation pay-off at* (L, D). In contrast, if Player 1 moves after observing Player 2, as in Figure 2(a), then as shown in Lemma 1, Player 1's incentive constraint implies that Player 2's continuation pay-off after (L, D) is at least $1 - R(z', \delta) - (1-\delta)/\delta$, *i.e.* Player 1's incentive to play D instead of U imposes a bound on the amount of punishment that Player 2 can expect after choosing L.

For our reputation result, we make extensive use of Lemma 1 in establishing the upper and lower bounds for Player 2's pay-offs (Lemmata 2 and 3). In Lemma 2, Player 2's pay-off is bounded along the equilibrium path. Consequently, in this lemma, the perfect information assumption is not required. Consider again the equilibrium described for the simultaneous-move game. The bound in Lemma 1 applies without alteration to the simultaneous-move game at node (R, D) (the node of interest for Lemma 2) because Player 1 believes that Player 2 plays R with probability one on the equilibrium path.

In contrast to Lemma 2, perfect information is essential for Lemma 3. In Lemma 3, we consider a strategy for Player 2 that plays L until Player 1 deviates from U, and we give a lower bound for Player 2's pay-off after (L, D). Lemma 1 provides a lower bound on Player 2's pay-off after (L, D) in the case of perfect information. However, there is no analog to Lemma 1 that provides a tight bound on Player 2's pay-off after (L, D) for the simultaneous-move game. For example, in the PBE, we construct that we can put no restrictions on pay-offs after node (L, D) beyond individual rationality and feasibility. This is because Player 1 expects to reach node (L, D) with probability zero.

4.2. *Without Assumption 1, the reputation result can fail*

Assumption 1 can fail in two ways. First, Assumption 1 fails if the pay-off profile where Player 1 receives $\bar{g}_1$ is not unique in G (*e.g.* if Γ is non-generic). Such a failure is depicted in Figure 10. Second, Assumption 1 fails if, $(\bar{g}_1, \hat{g}_2) \in G$, but Γ is not a strictly conflicting interests game. Such a failure is depicted in Figure 10(b). Below, we demonstrate that a reputation result can fail to obtain in these examples.



(a) A nongeneric common interest game. (b) The product choice game.

FIGURE 10
Games that fail to satisfy Assumption 1

In the non-generic common interest game depicted in Figure 10, suppose that the Stackelberg type of Player 1 always plays U and $\mu(S) < 1/2$. We describe a PBE where Player 1 receives a pay-off strictly lower than one. Suppose on the equilibrium path, (R, U) is played in the first K periods and (L, U) is played thereafter. Player 1 does not build a reputation in this PBE. Choose K such that both players receive a pay-off equal to $1/2$. Suppose that if Player 2 deviates from equilibrium by playing L, then Player 1's normal type reveals rationality by playing D, and the stage game equilibrium (L, D) is played thereafter. Consequently, Player 2 receives $\mu(S)$ if she deviates from the equilibrium strategy, which is less than her equilibrium pay-off $1/2$.

In the product choice game depicted in Figure 10(b), Player 1's dynamic Stackelberg pay-off is 1.5 and Player 2's minimax value is 0.[34] Although a dynamic Stackelberg strategy does not exist in this game, there are strategies that deliver a pay-off arbitrarily close to the dynamic Stackelberg pay-off. Suppose that Player 1's mixed actions are observed at the end of each period. One might conjecture that Player 1 can obtain a pay-off arbitrarily close to the dynamic Stackelberg pay-off by mimicking a type, ω^*, that plays H with probability $1/2+\epsilon$. However, this is not the case: Suppose that on the equilibrium path, Player 1 plays H with probability $1/2+\epsilon$, in each period. Player 2 plays N for the first K periods and plays B thereafter. Choose K such that $\delta^K = 1/2$. Consequently, no reputation is built on the equilibrium path and equilibrium pay-offs are $((1.5-\epsilon)/2, \epsilon/2)$. If Player 1 deviates from equilibrium and reveals rationality, then Player 2 plays N forever. If Player 2 deviates from equilibrium and plays B, then Player 1 reveals rationality by playing L. In the subsequent complete information game, an equilibrium with pay-offs $(1.5, 0)$ is played.[35] This construction is a PBE for any choice of ϵ if $\mu(\omega^*) < 1/2$: If Player 2 deviates and plays B, then she is facing ω^* with probability $\mu(\omega^*)$ and receives pay-off equal to ϵ. Alternatively, she is facing the normal type with probability $1-\mu(\omega^*)$ and receives pay-off equal to zero. However, $\mu(\omega^*)\epsilon < \epsilon/2$.

4.3. *The Stackelberg type*

In the repeated games that we consider here, the dynamic Stackelberg strategy is not necessarily unique. For example, in the game depicted in Figure 5, the grim-trigger strategy is also a dynamic Stackelberg strategy. Mimicking the grim-trigger strategy would, however, not give Player 1 a high pay-off. This is because the punishment phase is also very costly for Player 1. In contrast, the particular Stackelberg type that we choose is not very costly to mimic since the punishment phase is short, *i.e.* n^p is chosen minimally. If we had chosen any other finite length $n > n^p$ for the punishment phase instead of n^p, our reputation result would still hold.

4.4. *Other commitment types*

As noted previously by Schmidt (1993), Celentani *et al.* (1996), or Evans and Thomas (1997), if there is a chance that Player 1 is a commitment type other than the Stackelberg type, then Player 1 may be unable to build a reputation. Previous work has addressed this issue by assuming that types are learned due to exogenous noise (Celentani *et al.*, 1996, or Aoyagi, 1996) by

34. In this stage game, the Stackelberg pay-off is also equal to 1.5 because, for any $\epsilon > 0$, Player 1 can guarantee a pay-off equal to $1.5-\epsilon$ by playing H with probability $1/2+\epsilon$. Yet, a Stackelberg action does not exist. The unique action profile that yields Player 1 a pay-off exactly equal to 1.5 has player one mixing between H and L with equal probability and Player 2 playing B. However, both B and N are best responses to Player 1's equal mixture and if Player 2 best responds by playing N instead of B, then Player 1's pay-off is equal zero. Therefore, Player 1 cannot guarantee 1.5 by committing to this mixed action, *i.e.* a Stackelberg action does not exist.

35. Playing (N, L) in each period is a PBE of the complete information repeated game. Consequently, the threat of switching to (N, L) can incentives a patient Player 1 to play H with probability $1/2$ in each period.

restricting the class of games (Schmidt, 1993) or by considering more complicated types (Evans and Thomas, 1997).

In the environment we consider, the presence of commitment types can also hinder Player 1 from building a reputation. A patient Player 2 may resist the Stackelberg type because she fears punishment or expects a reward for not best responding, either from another commitment type or from Player 1's normal type. Accordingly, our reputation result holds because, as we show, punishments or rewards cannot come from Player 1's normal type; and because we assume that the probability of another commitment type is small compared to the probability of the Stackelberg type.

The restriction on the relative likelihood of other commitment types can be relaxed if the other commitment types are *uniformly learnable*. A uniformly learnable type reveals itself not to be the Stackelberg type at a rate that is bounded away from zero, uniformly across all histories. If the other commitment types are uniformly learnable, then Player 1 can play according to $\sigma_1(S)$, thereby ensuring that Player 2's posterior belief that Player 1 is a type in Ω_- is arbitrarily small in finitely many periods. If Player 2's posterior belief that Player 1 is a type in Ω_- is small, then Theorem 1 implies that Player 1's pay-off is close to one for sufficiently large discount factors. However, the restriction to uniformly learnable types is a non-trivial assumption. For example, it rules out the "perverse" type (see Schmidt, 1993) who plays like the Stackelberg type on the equilibrium path but responds to deviations in a history-dependent way.

In previous work, Schmidt (1993) and Celentani *et al*. (1996) establish reputation results with a non-myopic Player 2, even when the set of commitment types is arbitrary. Celentani *et al*. (1996) assume that Player 2's moves are imperfectly observed with full support.[36] This assumption ensures that all relevant histories are sampled with positive probability without any experimentation by Player 2. If Player 2's moves are imperfectly observed, then a rich set of commitment types is uniformly learnable. A similar assumption would also enable us to allow for a rich set of commitment types in the framework that we consider here.[37]

The reputation result of Schmidt (1993) obtains if there are conflicting interests in the stage game, Player 2's discount factor is fixed, and Player 1 is arbitrarily more patient. Conflicting interests imply that the punishment that Player 2 can expect from any other commitment type (her minimax pay-off) is no worse than best responding to the Stackelberg type and receiving her minimax pay-off. A commitment type may also reward Player 2 for not best responding to the Stackelberg type. But since Player 2's discount factor is fixed, a reward for Player 2 must entail behaviour, that differs from the Stackelberg type and that occurs in a bounded number of periods T. If Player 1 is sufficiently patient, then he can mimic the Stackelberg type for these T periods, depriving Player 2 of the reward and thus building a reputation. However, rewards for an equally patient Player 2 need not accrue in a bounded number of periods. A commitment type that rewards Player 2 for resisting the Stackelberg type in a history-dependent manner can hinder Player 1 from building a reputation against an equally patient opponent, even with SCI.

4.5. *Two-sided incomplete information*

The reputation results in games with asymmetric discounting are robust to the introduction of two-sided uncertainty, while the reputation result that we present in this paper is not. In order to obtain our one-sided reputation result, we allow for only one-sided uncertainty. In other words,

36. Also, see Aoyagi (1996) for a similar assumption.

37. See Atakan and Ekmekci (2008*a*), which assumes Player 2's moves are imperfectly observed with full support; under this assumption, it shows that the set of other types can be taken as the set of all finite automata and the perfect information assumption can be dropped.

we replace asymmetric discount factors as in Fudenberg and Levine (1989, 1992), or Celentani *et al*. (1996), with one-sided asymmetric information.

In a related paper, Atakan and Ekmekci (2008*b*), we consider a repeated game of perfect information with equally patient agents, two-sided LNCI or SCI, and two-sided uncertainty. In this related paper, we show two results: First, the repeated game has a unique equilibrium if the players are sufficiently patient. Second, under certain additional conditions, in the unique equilibrium of the repeated game, a war-of-attrition (similar to Abreu and Gul, 2000) is played prior to one player revealing herself to be the normal type, and once this has occurred, an equilibrium of the game of one-sided incomplete information, as characterized in Theorem 1, is played.

4.6. *Simultaneous-move games with SCI*

Cripps, Dekel and Pesendorfer (2005) obtain a reputation result for the Bayes–Nash equilibria of repeated simultaneous-move games with SCI. A similar result can be obtained using the method we develop here. In particular, redefine $R(z,\delta)$ using Bayes–Nash equilibrium instead of PBE. The upper bound established in Lemma A3 remains valid for Bayes–Nash equilibria. This is because all the arguments were constructed on the equilibrium path without any appeal to perfect information or sequential rationality. Also, $U_2(\sigma) \geq \hat{g}_2 = 0$ in any Bayes–Nash equilibrium. Consequently, functional inequality (6) holds, and a reputation result follows.

4.7. *Reputation in dynamic games*

We do not know whether our reputation result extends to more general dynamic games where a different stage game is played in each period. However, in the following restricted class of dynamic games, our reputation result also holds: any one of a finite number of stage games of perfect information is played in each period. All these stage games satisfy Assumption 1. The stage game that is played in a particular period is determined by a transition function, the transitions are stationary, and the transitions depend only on which game was played in the previous period, but not on the outcome of the game played in the previous period. For example, if the battle-of-the-sexes game in Figure 1 is played in the odd periods and if the battle-of-the-sexes game in Figure 2(d) is played in the even periods, then our reputation result would hold.

APPENDIX

Proof of Theorem 1.

Normalize pay-offs, without loss of generality, such that

$$\bar{g}_1 = 1; \quad g_1(a_1,a_2) \geq 0 \quad \text{for all} \quad a \in A; \quad \text{and} \quad g_2(a_1^s, a_2^b) = 0. \tag{A.1}$$

Recall that $M = \max\{\max\{|g_1|,|g_2|\} : (g_1,g_2) \in F\}$, hence $M \geq 1$.

For any $z \in (0,1]$, let $K(z) = \max\{\frac{4\rho}{zl}, \frac{8M}{zl}(\rho n^p + 2), 2\}$. For any $z \in (0,1]$, let

$$f(z) = K(z)^{\bar{n}(z)}, \tag{A.2}$$

where $\bar{n}(z)$ is the smallest positive integer j such that $(1 - zl/4\rho)^{j-1} < z$. Note that both K and $\bar{n}$ are decreasing, positive-valued functions of z. Hence, $f : (0,1] \to \mathbb{R}^{++}$ is a decreasing positive-valued function.

In what follows, *we fix constant* $\underline{z} > 0$, and *we fix constants*

$$K = K(\underline{z}) \quad \text{and} \quad \bar{n} = \bar{n}(\underline{z}). \tag{A.3}$$

Also, *we fix constant* $\phi \in [0,1)$. We show that for any $\mu \in \Delta(\Omega)$ such that $\mu(S) \geq \underline{z}$ and $\mu(\Omega_-)/\mu(S) \leq \phi$, and for any PBE strategy profile σ of $\Gamma^\infty(\mu,\delta)$, the following inequality holds

$$U_1(\sigma,\delta) \geq 1 - f(\underline{z})\max\{1-\delta,\phi\} \geq 1 - K^{\bar{n}}\max\{1-\delta, \mu(\Omega_-)/\mu(S)\}.$$

Lemma A1. *Posit perfect information and Assumption 1. There exists $\delta^* \in [0,1)$ and $l > 0$ such that for any $r \geq 0$, if $U_1(\sigma_1(S), \sigma_2, \delta) = 1 - r$, then $U_2(\sigma_1(S), \sigma_2, \delta) \leq -lr$, for all $\delta > \delta^*$.*

Proof. The definition of n^p given in inequality (1) implies that there exists a $\delta^* < 1$ and $l > 0$ such that, for all $\delta > \delta^*$, and for any $a_2 \in A_2$ such that $g_1(a_1^s, a_2) < 1$ and any $a_2' \in A_2$, we have

$$g_2(a_1^s, a_2) + \sum_{k=1}^{n^p-1} \delta^k g_2(a_1^p, a_2') < -ln^p. \tag{A.4}$$

For public history $h^t = (y^0, y^1, \ldots, y^t)$, let $i(h^t) = 1$, if $g_1(y^t) < 1$ and $\sigma_1(S, h^t) = a_1^s$; and $i(h^t) = 0$, otherwise. Player 1 receives at least zero in any period t where $i(h^t) = 1$ and also receives at least zero in the subsequent $n^p - 1$ period punishment phase. In all other periods, Player 1 receives one. Consequently,

$$U_1(\sigma_1(S), \sigma_2, \delta) \geq 1 - n^p(1-\delta)\mathbb{E}_{(\sigma_1(S),\sigma_2)}\left[\sum_{t=0}^{\infty} \delta^t i(h^t)\right],$$

and $(1-\delta)\mathbb{E}_{(\sigma_1(S),\sigma_2)}\left[\sum_{t=0}^{\infty} \delta^t i(h^t)\right] \geq r/n^p$.[38] If $i(h^t) = 1$, then Player 2 receives a total discounted pay-off of at most $-n^p l(1-\delta)$ for periods t through $t + n^p - 1$, if $\delta > \delta^*$ by inequality (A.4). In any period where a_1^s is played and $i(h^t)$=0, Player 2 receives zero. Consequently,

$$U_2(\sigma_1(S), \sigma_2) \leq -n^p l(1-\delta)\mathbb{E}_{(\sigma_1(S),\sigma_2)}\left[\sum_{t=0}^{\infty} \delta^t i(h^t)\right] \leq -lr,$$

if $\delta > \delta^*$. ‖

Remarks A1. *We argue that $U_1^C(\sigma_1(S), \delta) = 1$, i.e. $\sigma_1(S)$ is a dynamic Stackelberg strategy, for all $\delta > \delta^*$. Lemma A1 implies that if $U_1(\sigma_1(S), \sigma_2, \delta) < 1$, then $U_2(\sigma_1(S), \sigma_2, \delta) < 0$, for all $\delta > \delta^*$. Thus, if $U_2(\sigma_1(S), \sigma_2, \delta) \geq 0$, then $U_1(\sigma_1(S), \sigma_2, \delta) \geq 1$, for all $\delta > \delta^*$. If Player 2 plays a_2^b in each period of the repeated game against $\sigma_1(S)$, then Player 2's pay-off is equal to zero. Therefore, if $\sigma_2 \in BR(\sigma_1(S), \delta)$, then $U_2(\sigma_1(S), \sigma_2, \delta) \geq 0$ and as a consequence $U_1(\sigma_1(S), \sigma_2, \delta) \geq 1$ for all $\delta > \delta^*$. Also, if $\sigma_2 \in BR(\sigma_1(S), \delta)$, then $U_2(\sigma_1(S), \sigma_2, \delta)$ is at least as large as Player 2's minimax. Hence, if $\sigma_2 \in BR(\sigma_1(S), \delta)$, then $U_1(\sigma_1(S), \sigma_2, \delta) = 1$ for all $\delta > \delta^*$. This follows because Player 1's highest pay-off compatible with Player 2's individual rationality is equal to one.*

In what follows, *we assume that* $\delta > \delta^*$, where δ^* is the cutoff established in Lemma A1.

Definition A1. *For any $z \in (0,1]$, define the maximal resistance function as follows:*

$$\bar{R}(z, \delta) = \sup\{R(\mu, \delta) : \mu \in \Delta, \mu(S) \geq z, \mu(\Omega_-)/\mu(S) \leq \phi\}^+,$$

where Δ is the set of all measures over $\Sigma_1 \cup \{N\}$ with countable support, each commitment type is identified by the strategy that it plays, and Ω is the support of μ.[39]

Remarks A2. *Definition A1 implies that $\bar{R}(\cdot, \delta) : (0,1] \to [0,1]$ is a non-increasing function.*

Lemma A2. *Suppose that $\mu(\Omega_-)/\mu(S) \leq \phi$. Pick any PBE σ of $\Gamma^\infty(\mu, \delta)$, and any period t public history $h = (h^t, d_0)$; and suppose Player 1 deviates from $\sigma_1(S)$ at node d_0 with positive probability. Let h^{t+1} be any public history of terminal nodes that is reached with positive probability under $\Pr_{(\sigma_1(N)|_h, \sigma_2|_h)}$; and let $h' = (h^t, d')$ be the public history that is reached immediately (with positive probability under $\Pr_{(\sigma_1(S)|_h, \sigma_2|_h)}$) if $\sigma_1(S)$ is used at d. For any $z' > 0$, if $\mu(S|h') \geq z'$, then*

$$|U_2(\sigma_1(N), \sigma_2, \delta|h^{t+1})| \leq \rho(\bar{R}(z', \delta) + n^p M(1-\delta)/\delta), \quad \textit{if } \Gamma \textit{ satisfies Assumption 1 } (i)\textit{; and}$$
$$U_2(\sigma_1(N), \sigma_2, \delta|h^{t+1}) \leq \rho(\bar{R}(z', \delta) + n^p M(1-\delta)/\delta), \quad \textit{if } \Gamma \textit{ satisfies Assumption 1 } (ii).$$

38. The bound on Player 1's pay-off is crude, especially for low δ.
39. For any $a \in \mathbb{R}$, $a^+ = \max\{a, 0\}$.

Proof. Note that Player 1's reputation level $\mu(S|h') \geq z'$ and $\mu(\Omega_-|h')/\mu(S|h') \leq \phi$. Therefore, if a history (h^k, d'') is consistent with $\sigma_1(S)$ and if the node specified by the history (h^k, d'') comes after the node specified by the history (h^t, d'), then Player 1's reputation level $\mu(S|h^k, d'') \geq z'$ and $\mu(\Omega_-|h^k, d'')/\mu(S|h^k, d'') \leq \phi$. If Player 1 plays according to $\sigma_1(S)$ at d_0 and through the remaining nodes of period t, then he obtains at least zero for the period and an $n^p - 1$ period punishment phase may ensue. His pay-off is at least zero in these periods. Consequently, if he plays according to $\sigma_1(S)$, his pay-off is at least

$$0 \times (1-\delta^{n^p}) + \delta^{n^p}(1-\bar{R}(z',\delta)) = \delta^{n^p}(1-\bar{R}(z',\delta))$$

because $\bar{R}$ is non-increasing. Alternatively, if he chooses a move that differs from the move that $\sigma_1(S)$ would have chosen, then he receives at most $M(1-\delta)$ for the period, and $U_1(\sigma, \delta|h^{t+1})$ as his continuation pay-off. Therefore, $M(1-\delta) + \delta U_1(\sigma, \delta|h^{t+1}) \geq \delta^{n^p}(1-\bar{R}(z',\delta))$. This implies

$$U_1(\sigma, \delta|h^{t+1}) \geq \delta^{n^p-1}(1-\bar{R}(z',\delta)) - M(1-\delta)/\delta \geq 1 - \bar{R}(z',\delta) - n^p M(1-\delta)/\delta,$$

where the last inequality follows because $M \geq 1$, by definition. The bounds on Player 2's pay-off follow from inequalities (2) and (3), and from the fact that the pay-off profile $(U_1(\sigma, \delta|h^{t+1}), U_2(\sigma_1(N), \sigma_2, \delta|h^{t+1}))$ is an element of the set F. ‖

Definition A2 (Stopping Time). *For any integer k, $E_{[0,k]}$ denotes the event (set of infinite public histories) where Player 1 deviates from $\sigma_1(S)$ for the first time in period t for some $0 \leq t \leq k$. For any strategy profile $\sigma = (\{\sigma_1(\omega)\}_{\omega\in\Omega}, \sigma_2)$, any measure $\mu \in \Delta$, and any $z' \in (0,1]$, let*

$$T(\sigma, \mu, z') = \min\{k : \mu(S) \geq z'(1-\pi(k))\},$$

where $\pi(k) = \sum_{\omega\in\Omega} \mu(\omega) \Pr_{(\sigma_1(\omega),\sigma_2)}[E_{[0,k]}]$; and let $T(\sigma, \mu, q) = \infty$ if the set is empty.

Suppose that Player 1's initial reputation level $\mu(S) = z$ and $\mu(\Omega_-)/\mu(S) \leq \phi$; pick $z' > 0$ and pick a strategy profile $\sigma^* = (\{\sigma_1(\omega)\}_{\omega\in\Omega}, \sigma_2^*)$. Let $T = T(\sigma^*, \mu, z')$. Further suppose that σ_2^* is a pure strategy. Because both $\sigma_1(S)$ and σ_2^* are pure strategies, there is a unique path of play that is induced by $\sigma_1(S)$ and σ_2^*. Suppose that $T < \infty$ and let h^T and h^{T+1} denote the unique public histories of terminal nodes consistent with $(\sigma_1(S), \sigma_2^*)$. If $z < z'$, then the stopping time definition and Bayes' rule implies that that $\mu(S|h^T) < z'$ and $\mu(S|h^{T+1}) \geq z'$. Therefore, there exists a unique public history (h^T, d^*) consistent with $(\sigma_1(S), \sigma_2^*)$ such that $\mu(S|h^T, d^*) < z'$ and $\mu(S|h') \geq z'$ where $h' = (h^T, d')$ is the public history that is reached immediately after d^* if $\sigma_1(S)$ is used at node d^* in period T. Also, by Bayes' rule, the total probability that Player 1 deviates from the Stackelberg strategy at any decision node (in periods 0 through T) up to but excluding (h^T, d^*) is at most $1 - z/z'$.

Lemma A3. *Posit perfect information and Assumption 1. For any $\mu \in \Delta$ such that $\mu(S) = z > 0$ and $\mu(\Omega_-)/\mu(S) \leq \phi$, pick a PBE σ of $\Gamma^\infty(\mu, \delta)$ such that $r(\delta, \sigma_2) \geq R(\mu, \delta) - \xi$.[40] For the chosen PBE σ and any $z' > 0$,*

$$U_2(\sigma, \delta) \leq \rho(q(z, z')\bar{R}(z, \delta) + \bar{R}(z', \delta) + 2n^p M\epsilon) + 5M\epsilon - (R(\mu, \delta) - \xi)zl, \tag{A.5}$$

where $\epsilon = \max\{\phi, 1-\delta\}$ and $q(z, z') = \max\{1 - z/z', 0\}$.

Proof. Choose a pure strategy σ_2^* in the support of the possibly mixed strategy σ_2 such that $r(\sigma_2^*, \delta) \geq R(\mu, \delta) - \xi$. Such a pure strategy exists because the mixed strategy σ_2 has resistance of at least $R(\mu, \delta) - \xi$. Let profile $\sigma^* = (\{\sigma_1(\omega)\}_{\omega\in\Omega}, \sigma_2^*)$ and let $T = T(\sigma^*, \mu, z')$. If $z < z'$ and $T < \infty$, then let (h^T, d^*) denote the unique public history consistent with $(\sigma_1(S), \sigma_2^*)$ such that $\mu(S|(h^T, d^*)) < z'$ and $\mu(S|h') \geq z'$ where $h' = (h^T, d')$ is the public history that is reached immediately after (h^T, d^*) if $\sigma_1(S)$ is used at node d^*. If $z' \leq z$, then $T = 0$ and we let d^* denote the initial node of the game. If $T = \infty$, then we say $d^* = \infty$ which means that there are no decision nodes that come after d^*.

Given that $\mu(S) = z$ and $\mu(\Omega_-)/\mu(S) \leq \phi$, if $h = (h^t, d'')$ is a public history that is consistent with $(\sigma_1(S), \sigma_2^*)$, then $\mu(\Omega_-|h)/\mu(S|h) \leq \phi$ and $\mu(S|h) \geq z$; and moreover, if the decision node (h^t, d'') comes after d^*, then $\mu(S|h) \geq z'$.

Let E_1 denote the event, *i.e.* set of infinite histories, where Player 1 deviates from $\sigma_1(S)$ in a decision node before (and excluding) the decision node d^* of period T. Also, let E_2 denote the event where Player 1 deviates from $\sigma_1(S)$ in

40. For each $\xi > 0$, such a PBE of $\Gamma^\infty(\mu, \delta)$ exists because the resistance function R is defined as the supremum over the set $\{r(\delta, \sigma_2) : \sigma_2$ is part of a PBE of $\Gamma^\infty(\mu, \delta)\}$.

a decision node after (and including) the decision node d^* of period T.[41] We will bound Player 2's pay-off from σ^* in the following five events: $\omega = N$ and E_1; $\omega = N$ and E_2; $\omega = N$ and Player 1 never deviates from $\sigma_1(S)$; $\omega = S$; and $\omega \in \Omega_-$.

Before proceeding to bound Player 2's pay-off in the five events, as a preliminary step, we argue that Player 2's pay-off until the period t where Player 1 deviates from $\sigma_1(S)$ for the first time is at most $(1-\delta)M \leq \epsilon M$. To see why, consider the following three possibilities: First, if Player 2 plays a_2^b in each period until time t, then her pay-off is zero. Second, if Player 2 deviates from a_2^b in period $t' \leq t - n^p$, then she receives at most $(1-\delta)M$ in period t' and a punishment phase ensues. Lemma A1 implies that Player 2's discounted pay-off, for periods t' through $t' + n^p - 1$, is negative. Third, if Player 2 deviates from a_2^b in period $t' < t$ but $t' > t - n^p$, then she receives at most $(1-\delta)M$ in period t', a punishment phase ensues (but is not completed before period t), and she receives at most zero in periods $t'+1$ through $t-1$, *i.e.* she receives at most zero in each period of the incomplete punishment phase.

We now bound Player 2's pay-off in the event $\omega = N$ and E_1. Suppose that $h^\infty \in E_1$, then let $h = (h^j, d)$ denote the node in period j in which Player 1 deviates from $\sigma_1(S)$ for the first time in the infinite public history h^∞. Player 2's pay-off until period j is at most ϵM and Player 2's pay-off in period j is at most ϵM. Lemma A2 and the fact that $\epsilon \geq (1-\delta)$ together imply that $U_2(\sigma_1(N), \sigma_2^*, \delta | h^{j+1}) \leq \rho(\bar{R}(z,\delta) + \epsilon M n^p/\delta)$. Hence, for any such period j, Player 2's repeated game pay-off is at most

$$M\epsilon + \delta^j M\epsilon + \delta^{j+1}\rho(\bar{R}(z,\delta) + \epsilon M n^p/\delta) \leq 2M\epsilon + \rho(\bar{R}(z,\delta) + n^p M\epsilon).$$

We therefore obtain

$$U_2(\sigma_1(N), \sigma_2^*, \delta | E_1) \leq 2M\epsilon + \rho(\bar{R}(z,\delta) + n^p M\epsilon). \tag{A.6}$$

We bound Player 2's pay-off in the event $\omega = N$ and E_2. Suppose that $h^\infty \in E_2$, then let $h = (h^j, d)$ denote the node at which Player 1 deviates from $\sigma_1(S)$ for the first time in the infinite public history h^∞. Player 1's reputation is at least z' if he plays according to $\sigma_1(S)$ at the decision node d of period j. Consequently, Lemma A2 implies that $U_2(\sigma_1(S), \sigma_2^*, \delta | h^{j+1}) \leq \rho(\bar{R}(z',\delta) + n^p M\epsilon/\delta)$. As a result, an argument identical to that in the previous paragraph implies that

$$U_2(\sigma_1(N), \sigma_2^*, \delta | E_2) \leq 2M\epsilon + \rho(\bar{R}(z',\delta) + n^p M\epsilon). \tag{A.7}$$

Player 2's pay-off in the event that $\omega = S$ (*i.e.* the event in which she faces type S) is at most $-(R(\mu,\delta) - \xi)l$, and the probability of this event is equal to z. This is because Player 2's resistance is at least $R(\mu,\delta) - \xi$ for the strategy σ_2^*; hence, she loses at least $(R(\mu,\delta) - \xi)l$ against S, by Lemma A1. Player 2's pay-off in the event that $\omega \in \Omega_-$ (*i.e.* the event in which she faces any other commitment type) is at most M, and the probability of this event is at most $\phi z \leq \phi \leq \epsilon$. A bound on Player 2's pay-off in the event that $\omega = N$ and E_1 is given by inequality (A.6), and the probability of this event is at most $q(z,z')$. A bound on Player 2's pay-off in the event that $\omega = N$ and E_2 is given by inequality (A.7), and the probability of this event is at most one. Player 2's pay-off in the event that $\omega = N$ and Player 1 never deviates from $\sigma_1(S)$ is at most zero. Consequently,

$$U_2(\sigma^*, \delta) \leq q(z,z')\rho\bar{R}(z,\delta) + \rho\bar{R}(z',\delta) - z(R(\mu,\delta) - \xi)l + 2\rho n^p M\epsilon + 5M\epsilon.$$

Since σ_2^* is in the support of PBE strategy σ_2, we have $U_2(\sigma,\delta) = U_2(\sigma^*,\delta)$. Hence,

$$U_2(\sigma,\delta) \leq q(z,z')\rho\bar{R}(z,\delta) + \rho\bar{R}(z',\delta) - z(R(\mu,\delta) - \xi)l + 2\rho n^p M\epsilon + 5M\epsilon.$$

‖

Lemma A4. *Posit perfect information and Assumption 1 (i). Suppose that $\mu(S) = z > 0$ and $\mu(\Omega_-)/\mu(S) \leq \phi$. In any PBE σ of $\Gamma^\infty(\mu,\delta)$ and for any $z' > 0$, we have*

$$U_2(\sigma,\delta) \geq -\rho(q(z,z')\bar{R}(z,\delta) + \bar{R}(z',\delta) + 2n^p M\epsilon) - 3M\epsilon, \tag{A.8}$$

where $\epsilon = \max\{\phi, 1-\delta\}$ and $q(z,z') = \max\{1 - z/z', 0\}$.

Proof. Fix a PBE profile σ of $\Gamma^\infty(\mu,\delta)$. Let σ_2^* denote a pure strategy that moves according to a_2^b after any public history h that is consistent with $\sigma_1(S)$; and coincides with a pure strategy in the support of the PBE strategy σ_2 if public history h is not consistent with $\sigma_1(S)$. Let profile $\sigma^* = (\{\sigma_1(\omega)\}_{\omega\in\Omega}, \sigma_2^*)$ and let $T = T(\sigma^*, \mu, z')$. If $z < z'$

41. Observe that if d^* is the initial node, then $E_1 = \emptyset$. Also, if $d^* = \infty$, then $E_2 = \emptyset$.

and $T < \infty$, then let (h^T, d^*) denote the unique public history consistent with $(\sigma_1(S), \sigma_2^*)$ such that $\mu(S|(h^T, d^*)) < z'$ and $\mu(S|h') \geq z'$ where $h' = (h^T, d')$ is the public history that is reached immediately after (h^T, d^*) if $\sigma_1(S)$ is used at node d^*. If $z' \leq z$, then $T = 0$ and we let d^* denote the initial node of the game. If $T = \infty$, then we say $d^* = \infty$ which means that there are no decision nodes that come after d^*.

Because (a_1^s, a_2^b) is played in each period under $(\sigma_1(S), \sigma_2^*)$, Player 2 receives zero in each period until Player 1 deviates from $\sigma_1(S)$. Also, Player 2's pay-off in the period in which Player 1 deviates from $\sigma_1(S)$ is at least $-M\epsilon$. Using the reasoning in Lemma A3 and applying Lemma A2, we obtain

$$U_2(\sigma_1(N), \sigma_2^*, \delta|E_1) \geq -\rho(\bar{R}(z,\delta) + n^p M\epsilon) - M\epsilon,$$

and

$$U_2(\sigma_1(N), \sigma_2^*, \delta|E_2) \geq -\rho(\bar{R}(z',\delta) + n^p M\epsilon) - M\epsilon,$$

where E_1 and E_2 are the events defined in Lemma A3.

If Player 1 never deviates from $\sigma_1(S)$, then Player 2 receives zero. Player 2 can get at least $-M$ against any other commitment type, whom she faces with probability of at most $\phi \leq \epsilon$; she gets zero against type S, whom she faces with probability z. Following the same reasoning as in Lemma A3 and because σ_2 is part of the PBE σ, we obtain

$$U_2(\sigma,\delta) \geq U_2(\sigma^*,\delta) \geq -\rho q(z,z')\bar{R}(z,\delta) - \rho\bar{R}(z',\delta) - 2\rho n^p M\epsilon - 3M\epsilon.$$

‖

Proof. [Completing the Proof of Theorem 1 by Using Lemmata A3 and A4]. If Γ satisfies Assumption 1 and perfect information, then inequality (A.5) is satisfied, by Lemma A3. If Γ satisfies Assumption 1(i) and perfect information, then inequality (A.8) is satisfied, by Lemma A4. Also, if Γ satisfies Assumption 1(ii), then $U_2(\sigma,\delta) \geq \hat{g}_2 = 0$, and inequality (A.8) is trivially satisfied because the right-hand side of the inequality is negative. By combining the upper and lower bounds for $U_2(\sigma,\delta)$, given by inequalities (A.5) and (A.8), and using the fact that $\xi > 0$ can be chosen arbitrarily, we obtain

$$zlR(\mu,\delta) \leq 2\rho(q(z,z')\bar{R}(z,\delta) + \bar{R}(z',\delta) + 2n^p M\epsilon) + 8M\epsilon, \tag{A.9}$$

for any $\mu \in \Delta$ such that $\mu(S) = z$ and $\mu(\Omega_-)/\mu(S) \leq \phi$, and for any $z' \in (0,1]$. Pick another measure $\mu' \in \Delta$ such that $\mu'(S) \geq z$ and $\mu'(\Omega_-)/\mu'(S) \leq \phi$. By rewriting inequality (A.9) for μ' and $z' \in (0,1]$ and by rearranging, we obtain the following inequality:

$$R(\mu',\delta) \leq (2\rho(q(\mu'(S),z')\bar{R}(\mu'(S),\delta) + \bar{R}(z',\delta) + 2n^p M\epsilon) + 8M\epsilon)/\mu'(S)l. \tag{A.10}$$

However, $q(z,z') \geq q(\mu'(S),z')$, because $\mu'(S) \geq z$; and $\bar{R}(z,\delta) \geq \bar{R}(\mu'(S),\delta) \geq 0$, because $\bar{R}$ is non-negative and non-increasing in z. Substituting z for $\mu'(S)$, $q(z,z')$ for $q(\mu'(S),z')$, and $\bar{R}(z,\delta)$ for $\bar{R}(\mu'(S),\delta)$ on the right-hand side of inequality (A.10) delivers the following:

$$zlR(\mu',\delta) \leq 2\rho(q(z,z')\bar{R}(z,\delta) + \bar{R}(z',\delta) + 2n^p M\epsilon) + 8M\epsilon, \tag{A.11}$$

for all $\mu' \in \Delta$ such that $\mu'(S) \geq z$ and $\mu'(\Omega_-)/\mu'(S) \leq \phi$. Because $\bar{R}(z,\delta)$ is the supremum over the set $\{R(\mu',\delta) : \mu \in \Delta, \mu'(S) \geq z$ and $\mu'(\Omega_-)/\mu'(S) \leq \phi\}$, and because each $R(\mu',\delta)$ in this set satisfies inequality (A.11), we obtain the following:

$$zl\bar{R}(z,\delta) \leq 2\rho(q(z,z')\bar{R}(z,\delta) + \bar{R}(z',\delta) + 2n^p M\epsilon) + 8M\epsilon. \tag{A.12}$$

For any $z \geq \underline{z}$, substituting $\underline{z}$ for z in inequality (A.12) and rearranging gives the following functional inequality:

$$\bar{R}(z,\delta)(\underline{z}l - 2\rho q(z,z')) \leq 2\rho\bar{R}(z',\delta) + 4M(\rho n^p + 2)\epsilon. \tag{A.13}$$

Let $\underline{q} = \underline{z}l/4\rho$. If $z, z' \in [\underline{z}, 1]$ and $z \in [z'(1-\underline{q}), z']$, then $q(z,z') \leq \underline{q}$. Hence, substituting $\underline{q}$ for $q(z,z')$ in inequality (A.13) we obtain the following:

$$\bar{R}(z,\delta)(\underline{z}l - 2\rho\underline{q}) \leq 2\rho\bar{R}(z',\delta) + 4M(\rho n^p + 2)\epsilon.$$

Substituting $\underline{z}l/4\rho$ for $\underline{q}$ in the previous inequality and rearranging, we obtain the following:

$$\bar{R}(z,\delta) \leq \frac{4\rho}{\underline{z}l}\bar{R}(z',\delta) + \frac{8M}{\underline{z}l}(\rho n^p + 2)\epsilon.$$

Using the fact that $\bar{R}(z', \delta) \geq 0$, and substituting $K = \max\{\frac{4\rho}{\underline{z}l}, \frac{8M}{\underline{z}l}(\rho n^p + 2), 2\}$ for $\frac{4\rho}{\underline{z}l}$ and $\frac{8M}{\underline{z}l}(\rho n^p + 2)$ in the previous inequality, we obtain the following:

$$\bar{R}(z, \delta) \leq K\bar{R}(z', \delta) + K\epsilon. \tag{A.14}$$

However, the functional inequality (A.14) is identical to inequality (7) (since K and ϵ in inequality (A.14) serve the same roles as C and $1-\delta$ in inequality (7)). Also, $\bar{R}(1, \delta) = 0$. Consequently, an argument identical to the one used to establish Corollary 1 implies that $\bar{R}(\underline{z}, \delta) \leq \sum_{j=1}^{\bar{n}-1} K^j \epsilon$, where $\bar{n}$ is the smallest integer j such that $(1-\underline{q})^{j-1} < \underline{z}$. Because $K \geq 2$, we have $\bar{R}(\underline{z}, \delta) \leq \sum_{j=1}^{\bar{n}-1} K^j \epsilon \leq K^{\bar{n}}\epsilon = K^{\bar{n}} \max\{1-\delta, \phi\}$. For any μ such that $\mu(S) \geq \underline{z}$ and $\mu(\Omega_-)/\mu(S) \leq \phi$, and for any PBE strategy σ of $\Gamma^\infty(\mu, \delta)$, we have $U_1(\sigma, \delta) \geq 1 - R(\mu, \delta)$ and $R(\mu, \delta) \leq \bar{R}(\underline{z}, \delta)$. Consequently, $U_1(\sigma, \delta) \geq 1 - K^{\bar{n}} \max\{1-\delta, \phi\}$. ‖

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