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Bargaining and Reputation in Search Markets

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This article considers a two-sided search market where firms and workers are paired to bargain over a unit surplus. The matching market serves as an endogenous outside option for the agents. The market includes inflexible commitment types who demand a constant portion of any match surplus. The frequency of such types is determined in equilibrium. An equilibrium where there are significant delays in reaching an agreement and where negotiations occasionally break down on the equilibrium path is constructed. Such an equilibrium exists and commitment types affect bargaining dynamics even if the equilibrium frequency of such types is negligible. If the inflows of firms and workers into the market are symmetric, then bargaining involves two-sided reputation building and reputation concerns lead to delays and inefficiency. Access to the market exacerbates bargaining inefficiencies caused by inflexible types. If the inflows of workers and firms are sufficiently asymmetric, then bargaining involves one-sided reputation and commitment types determine the terms of trade.

Key words: Bargaining, Reputation, Search, Dynamic Matching, War-of-Attrition

JEL Codes: C78, D83

1. INTRODUCTION AND RELATED LITERATURE

In certain search markets, individuals or institutions determine the terms at which they trade through bilateral bargaining. The labour and housing markets are two centrally important examples of such markets. Consider a worker bargaining with a firm over the terms of an employment contract. While negotiating, the firm can remain intransigent or can choose to meet the worker's remuneration expectation. Appearing inflexible is potentially valuable for the firm because it can force the worker into accepting a lower remuneration. However, inflexibility can also delay reaching an agreement or even lead to a breakdown of bargaining. The firm's benefit from intransigent will depend on (1) how effectively the firm is able to convince the worker that the firm is indeed inflexible (*i.e.* how likely it is that the firm is inflexible) and on (2) the worker's cost of generating another employment opportunity with a more accommodating firm (*i.e.* the worker's outside option). Similar concerns are also pertinent for the worker who needs to determine his own bargaining posture. However, outside options are typically determined by market conditions. Also, the probability that a worker meets a firm which is inflexible depends on whether there are many firms which are indeed inflexible and whether firms, who are privately informed about their flexibility, adopt inflexible bargaining postures.

Motivated by the interdependence between bargaining posture, bargaining outcomes, and market conditions, we analyse a two-sided search market in which the outside options and the distribution of types are endogenously determined and each player is uncertain about his bargaining partner's type. In each period, a constant measure of workers and firms enters the market, where the inflow of firms is at least as large as the inflow of workers. Workers and firms are matched to bargain over a unit surplus. The matching market serves as the endogenous outside option for agents in a bargaining relationship. Workers and firms exit the market through successfully agreeing on a wage for the worker, or due to some exogenous factor, such as the firm going bankrupt, or the worker retiring. A fraction of the entering population on each side is composed of inflexible types who are committed to demanding a constant portion of any match surplus.[1] The steady-state frequency of such types in the market is determined in equilibrium.

We focus on the steady-state equilibria of such a market.[2] This restriction captures the intuition that agents who have the same preferences employ the same strategies, regardless of the calendar time in which they join the market. In our main result (Theorem 1), we show that a certain type of equilibrium, which we term an equilibrium with selective breakups (SBU equilibrium), always exists. In SBU equilibria, the bargainers either immediately settle on an agreement or they mimic the inflexible types to build a reputation and opt-out only after negotiating without an agreement for a payoff relevant length of time. Moreover, if the measure of the flexible (or uncommitted) firms that enter the market is less than the total measure of workers that enter the market, then such an equilibrium always exhibit delays in reaching an agreement in the bargaining stage and break-ups which occur at the end of negotiations. Therefore, the bargaining is inefficient.

Anecdotal evidence suggests that bilateral negotiations take time and occasionally end without an agreement because, one side (say the worker) prefers to quit negotiating rather than yield to his opponent's demands (*i.e.* the worker opts-out). Moreover, negotiation can end without an agreement even after significant time is spent at the bargaining table.[3] SBU equilibria, which were shown to exist in our main result, exhibit such bargaining dynamics. In SBU equilibria, the amount of time spent at the bargaining table before reaching an agreement is related to the magnitude of uncertainty surrounding the negotiation (*i.e.* the incomplete information about the agents' types). In turn, the distribution of types is endogenously determined in equilibrium and is related to the frequency of break-ups.

In our second result, we show that delays and therefore bargaining inefficiencies persist in SBU equilibria even if the measure of inflexible types entering the market is arbitrarily small and the market is composed of an equal number of firms and workers (Theorem 2).[4] Moreover, the probability with which negotiating parties reach an agreement is arbitrarily close to one and the endogenous fraction of inflexible types in the market is close to zero. Therefore, even an arbitrarily small fraction of inflexible types is sufficient to generate significant delays in reaching an agreement during bargaining when the outside options and type distributions are endogenously determined in a search market.

Our third result characterizes bargaining dynamics of SBU equilibria under two distinct market configurations when the search costs and exit rates are small (Theorem 3). First, if the total inflow

1. For example, a firm could be committed to a certain wage level for internal fairness considerations or due to contracts signed before they entered the search market.

2. Our restriction on the steady-state equilibria is restrictive, however is commonly used in the search literature. See for example, Shimer and Smith (2003), or the discussions in Shimer and Smith (2001) and Lauermann (Forthcoming) for a more detailed discussion and motivation for the steady-state restriction.

3. See for example Kennan and Wilson (1993) who discuss instances where protracted labour negotiations end without an agreement.

4. Note that our finding stands in stark contrast to Abreu and Gul (2000) who show, in a related model, that agents agree immediately at the limit without any commitment types.

of workers exceeds the inflow of flexible firms, then there are significant delays in reaching an agreement in the bargaining stage and the inefficiency in bargaining depresses outside options for both the workers and the firms. In such equilibria, firms and workers mimic inflexible types in anticipation of a better deal (*i.e.* there is two-sided reputation building) and terminate their bargaining relationship to search for a new partner if advantageous terms are not forthcoming. Second, if the inflow of workers is less than the inflow of flexible firms, *i.e.* if the workers are scarce, then there is an equilibrium in which the value of the workers' outside option is high and the firms' is low. In this equilibrium, the pair either settles on the inflexible demand of a worker immediately (*i.e.* there is one-sided reputation building) or the worker opts-out to search for a new bargaining partner who will acquiesce to the workers' inflexible demand. In such equilibria, bargaining is efficient and negotiations terminate (either successfully or unsuccessfully) without any delay.

A key takeaway from our results is as follows: the mere presence of inflexible types affect both bargaining dynamics and market outcomes. However, the exact nature of this effect depends on the inflow rates of workers and firms, *i.e.* the relative scarcity of firms and workers. If the inflow rates of the workers and firms are symmetric, *i.e.* if neither side is scarce, then the temptation to build a reputation results in two-sided reputation building, inefficient bargaining dynamics due to delays in reaching an agreement, and break-ups. Such equilibrium dynamics are sustained even when the inflexible types are arbitrarily rare. On the other hand, if the inflow rates are sufficiently asymmetric, *i.e.* if one side is relatively scarce, then there is one-sided reputation building and efficiency is restored in the bargaining stage: bargaining ends immediately with the scarce side receiving its inflexible demand. Thus, our results shed light on how uncertainty about bargaining postures can cause delays and break-ups in negotiations when such negotiations take place in a market context.

1.1. *Related literature*

The closest papers to ours are Compte and Jehiel (2002) and Abreu and Gul (2000). These papers analyse a bilateral bargaining model in which there are both flexible and inflexible types. Both papers consider the case where the outside option of a flexible type is less than the share offered by an inflexible opponent.[5] Equilibrium dynamics in this model resemble the unique equilibrium of a war of attrition in which the party who first reveals that he is flexible reaches an agreement at the terms of the inflexible opponent. The bargaining outcome is inefficient because agreements are delayed. Compte and Jehiel (2002) also analysed the case where the outside options of the flexible types exceed the share offered by an inflexible opponent. In this case, the presence of inflexible types has no impact on bargaining outcomes. The flexible types reach an agreement immediately at the terms predicated by the Rubinstein bargaining outcome.

In Compte and Jehiel (2002) and Abreu and Gul (2000), the outside options and the distribution of inflexible types are given exogenously. We build on their work by analysing a model where outside options as well as the distribution of types are determined endogenously in equilibrium. This model allows us to make three novel contributions to this line of research: first, we show how aggregate forces and the incentive to build a reputation interact to determine bargaining behaviour. If the inflow of workers and firms is symmetric, then bargaining is characterized by two-sided reputation building whereas if the inflows are sufficiently asymmetric, then there is an equilibrium with one-sided reputation building. Second, under the more interesting

5. Myerson (1991) proved a one-sided reputation result in bargaining games which was later used by Kambe (1999), Abreu and Gul (2000), Compte and Jehiel (2002), and Abreu and Pearce (2007) to prove two-sided reputation results.

case where the inflows of the two sides are symmetric, we show that agreements are reached after substantial delays and bargaining occasionally ends without an agreement even after substantial amount of time has been spent negotiating. This feature of our equilibrium is frequently observed in practice but absent in Abreu and Gul (2000) and Compte and Jehiel (2002) because agents do not have an outside option in the former and because outside options preclude delays and reputation effects in the latter. Third, again with symmetric inflows, we show that the delay in reaching an agreement remains substantial even as the inflow rate of inflexible types converges to zero, the probability that the parties eventually reach an agreement goes to one, and the steady-state frequency of inflexible types present in the market converges to zero. This contrasts with Abreu and Gul (2000) who established that the expected delay goes to zero as the fraction of inflexible types becomes arbitrarily small.

The main intuition for why our findings differ from those of Abreu and Gul (2000) and Compte and Jehiel (2002) is as follows: in our model, a player's outside option cannot exceed the demand of an inflexible opponent if the inflow rates are sufficiently symmetric. In addition, the outside option of one of the two flexible agents is exactly equal to the demand of her inflexible opponent, *i.e.* the seemingly knife-edge configuration of outside options not analysed by Compte and Jehiel (2002) arises as an equilibrium phenomenon. Because one of the agents is indifferent between yielding to an inflexible opponent and opting out, such an agent can and does terminate her bargaining relationship once she is convinced that her opponent is inflexible. Also, for such a configuration of outside options to remain possible even as the inflow rate of inflexible types converges to zero, bargaining must remain inefficient, *i.e.* there must be substantial delay in reaching an agreement. This configuration of outside options is the only configuration which is compatible with an endogenously determined steady state if the inflow rates are symmetric. This is because access to the market precludes outside options that are too high or too low. To see why, suppose that the outside option of, for example, a worker, is higher than the demand of an inflexible firm. Then, such a worker would never trade with an inflexible firm and thus inflexible firms, who never trade, would be much more frequently encountered than flexible ones. Hence, the amount of costly time such a worker spends in locating a flexible firm would push his outside option below what he could get by yielding but this is incompatible with a high outside option. On the other hand, if both flexible players' outside options are strictly less than the demands of inflexible opponents, then Abreu and Gul (2000)'s analysis implies that the strong player's expected payoff in the bargaining game must exceed what he can get by yielding. However, if search costs are small, then his outside option is nearly equal to his bargaining game payoff which is incompatible with a low outside option.

There are recent papers that analyse the role of outside options and incomplete information in a bargaining context. Lee and Liu (Forthcoming) study a repeated bargaining model where the exogenous outside option of a long-run player is unknown, and the short-run opponents gradually learn the outside option of the long-run player. In their setup there is one long-lived player, and the short lived players partially observe what happened in previous stages. In our setup, all players share the same discount factor, and the actions of a player in previous matches are not observed publicly. Özyurt (2011) studies a model in which a single buyer negotiates with two sellers who can be inflexible. In his model, the buyer's negotiations are publicly observed, hence he builds a reputation in the negotiations with one seller to capitalize in his negotiation with the second seller. In our setup, there are multiple buyers and sellers, and the actions within a relationship are private information of the agents in the relationship.

Our article is closely related to the search literature with anonymous matching with incomplete information. Wolinsky (1990) studies a decentralized matching model in which there is aggregate uncertainty about an unknown state of the world, and each agent has a piece of information. He shows that if agents meet through pairwise meetings, then even when the search frictions disappear, they trade without the full knowledge about the realization of the state of the world. Our setup is similar to his setup, however in our model there is no aggregate uncertainty. On the contrary, each agent's preference is his private information. Lauermann and Wolinsky (2008) analyse conditions under which a seller that has private information about the quality of his good can charge a high price when the good is high quality, if he can search from a large set of buyers.[6] Unlike in our model, the seller's information is correlated with the buyer's valuation for the object.

All the proofs are provided in the Appendix.

2. THE MODEL

In each instant, a unit measure of workers seeking jobs (side 1) and a measure $l \geq 1$ of firms trying to fill vacancies (side 2) enter a matching market. Each worker aims to find a job and each firm has one vacancy that it aims to fill. A fraction z_1 of the workers entering the market are commitment types and the remaining $1-z_1$ are fully flexible agents unencumbered by commitments. Likewise a proportion z_2 of entering firms are commitment types. In what follows, we will refer to a worker (firm) unencumbered by commitments as a flexible worker (or firm). Otherwise we specify that the agent is a *commitment type* by referring to them as an inflexible worker (or firm). Flexible workers and firms are standard utility maximizing agents. Inflexible workers and firms are standard *commitment types* as in Abreu and Gul (2000) and Compte and Jehiel (2002), or more generally as found in the reputations literature. We detail our behavioural assumptions in section 2.1 below.

In each instant of time, a portion of the unmatched agents in the market are randomly paired with a potential trading partner from the opposite side of the market to play a bilateral bargaining game which we describe in detail further below. A unit surplus is available for division between the paired agents in the bargaining game. The unit surplus represents the lifetime value created from a match between a worker and a firm. Agents only receive utility if they can agree on the division of the surplus. If two matched agents agree on the division of the surplus, then they form a successful match. Agents that form a successful match permanently leave the market.

The bargaining game can terminate without an agreement because an agent voluntarily opts-out. If the bargaining game terminates without an agreement, then both agents return to the unmatched population after $\tau > 0$ units of time. The parameter τ measures the amount of time it takes to generate a new bargaining opportunity and is a proxy for the magnitude of search frictions. If $\tau = 0$, then agents are able to generate new bargaining partners instantaneously.

We assume that the fraction $\delta := 1 - e^{-\rho\tau}$ of the agents in the unmatched population leave the market without trading due to exogenous reasons, where the constant $\rho > 0$ is the exogenously given death (or exit) rate per unit time. The agents who leave the market without trading receive a payoff equal to zero.[7]

In what follows we assume that agents have perfect recall. Moreover, an agent does not observe his bargaining partner's actions in her past relationships. Agents observe the actions of their bargaining partner while they bargain.

6. There are many other papers that analyse various markets using search methodology, such as Rubinstein and Wolinsky (1985, 1990), Samuelson (1992) and Serrano and Yosha (1993) as our paper does. A more complete overview can be found in Osborne and Rubinstein (1990). Also, see Merlo and Wilson (1995) for when delays can occur in bargaining when the surpluss is stochastic.

7. The exogenous death rate is a device we employ to ensure that the matching market remains in steady state. This modelling device has also been used by Shimer and Smith (2003), Shimer and Smith (2001), and Lauermann (Forthcoming).

2.1. *Agent types*

The labour market is composed of flexible and inflexible workers and firms. Flexible workers and firms are impatient with instantaneous rate of time preference r_1 and r_2, respectively. Consequently, if flexible agent i reaches an agreement that gives him y units of surplus after s units of time, then his utility is equal to $ye^{-r_i s}$.

Inflexible workers and firms are behavioural agents who insist on a θ_i share of the unit surplus and reject any offer that delivers less than θ_i. We assume that the demands of inflexible workers and firms are incompatible, that is, $\theta_1+\theta_2>1$. An inflexible player never opts-out as long as there is a positive probability that his opponent is not inflexible, and immediately opts-out otherwise. Consequently, the probability that two inflexible players remain in a bargaining relationship forever is equal to zero.[8]

An inflexible firm may be locked into asking for an inflexible share of the surplus because of inter firm wage equity concerns, previous contracts, or search committee dynamics. Similarly, an inflexible worker may be locked into an inflexible share because of the worker's expectation to match his previous wage. We do not explicitly model why the inflexible agents are inflexible in their demands but use the behavioural assumptions that we outlined in the previous paragraph to model their behaviour on the market.

2.2. *The bargaining game*

The bargaining game that we study is an alternating offers bargaining game with the possibility of opting out: The two players alternate in making offers and the time between offers is denoted by $\Delta \geq 0$. If $\Delta = 0$, then the game is a continuous time war of attrition in which the players have the option to opt-out. We describe this game in the paragraph below. If $\Delta > 0$, then the game is a discrete-time alternating-offers bargaining game in which the players have an option to opt-out. We describe this game in Appendix B. Moreover, we justify our use of a continuous time war of attrition to represent a market with frictionless bargaining (*i.e.* $\Delta = 0$) by arguing that the equilibria of the discrete-time alternating offers bargaining game converge to the war of attrition as the time between offers, Δ, shrinks to zero.[9] We focus the discussion in the main text on the war of attrition and the case where $\Delta = 0$ because it simplifies the exposition and allows us to explicitly derive equilibrium outcomes.

In the *continuous time war of attrition*, each player chooses whether to yield, insist or opt-out at each time t.[10] Therefore, a pure strategy for either player can be represented as a choice of a real time $t \in [0,\infty]$ and action $a \in \{Yield, Opt-out\}$. If the player chooses $t=\infty$ this represents that the player never yields or opts-out. The inflexible player never yields to an opponent. The inflexible player opts-out only after his opponent is known with certainty to also be an inflexible player.

If player i yields at time t before player j yields or opts-out, then flexible players i and j receive $(1-\theta_j)e^{-r_i t}$ and $\theta_j e^{-r_j t}$, respectively. If both players yield at the same time t, then flexible players i and j receive $e^{-r_i t}(\frac{\theta_i+1-\theta_j}{2})$ and $e^{-r_j t}(\frac{\theta_j+1-\theta_i}{2})$, respectively. If player i yields and player j opts-out at time t, then we assume that the players trade and flexible players i and j receive $(1-\theta_j)e^{-r_i t}$

8. We make this assumption only for tractability reasons. Alternatively, if we had assumed that when two inflexible types bargain, they never go to the pool of unmatched agents, the steady-state calculations would change, however all of the qualitative features of our results would continue to be true.

9. This result is now standard in the reputations literature see for instance Abreu and Gul (2000), Compte and Jehiel (2002) or Atakan and Ekmekci (2013) for similar findings.

10. Sometimes the phrases "reveal rationality" or "concede" are used to refer to the action "yield".

and $\theta_j e^{-r_j t}$, respectively. If player i opts-out at time t before player j yields, then the flexible players receive their outside options v_1 and v_2.

A mixed strategy σ_i for player i in this bargaining game is summarized by two cumulative distribution functions F_i and α_i such that $F_i(\infty)+\alpha_i(\infty)\leq 1$. The function $F_i(t)$ gives the total probability with which either the flexible or the inflexible player i yields at or before time t while the function $\alpha_i(t)$ gives the probability with which either the flexible or the inflexible player i opts-out at or before time t.

2.3. *The steady-state pool of unmatched agents and matching*

We denote the steady-state measure of unmatched workers and firms by W and F, respectively; the steady-state proportion of unmatched workers and firms who are inflexible (or committed) by c_1 and c_2, respectively; and the steady-state proportion of unmatched workers and firms who are flexible by $n_1=1-c_1$ and $n_2=1-c_2$. We let $m=W/F$ denote the market tightness parameter and we let $m_1=\min\{1,1/m\}$ and $m_2=\min\{1,m\}$. In steady state, an agent i is matched with a bargaining partner with probability m_i in any given period. In what follows, we refer to the vector $(W,c_i,m)_{i\in\{1,2\}}$ as the steady-state distribution.

The pool of unmatched agents available to be matched in period s is composed of agents that entered the market in period s; agents who did not exit (as a result of the exogenously given exit rate) and whose bargaining arrangement dissolved as a result of an opt-out in period $s-\tau$; and agents who in period $s-\tau$ were not paired with a bargaining partner and who did not exit. A worker is paired with a flexible firm with probability n_2 and with an inflexible firm with probability c_2. Consequently, measure $m_1Wn_1n_2=m_2Fn_1n_2$ of flexible worker and firm pairs; measure $m_1Wc_1n_2=m_2Fc_1n_2$ of inflexible worker and flexible firm pairs; measure $m_1Wn_1c_2=m_2Fn_1c_2$ of flexible worker and inflexible firm pairs; and measure $m_1Wc_1c_2=m_2Fc_1c_2$ of inflexible worker, inflexible firm pairs are created in each period.

Let us focus on a pair of agents who are paired in a given period. For this pair, let p denote the probability that they eventually consummate their match if both agents are flexible. Let q_1 denote the probability that they consummate their match if the firm is flexible and if the worker is inflexible. And, let q_2 denote the probability that they eventually consummate their match if the worker is flexible and if the firm is inflexible. These trading probabilities are uniquely defined by any symmetric strategy profile σ. Also, we denote the probability that a pair of agents eventually trade given that they are matched by $\pi=n_1n_2p+n_1c_2q_2+c_1n_2q_1$ and we let $c=(c_1,c_2)$, $q=(q_1,q_2)$ and $z=(z_1,z_2)$.

In a steady state, the measure of agents leaving the market in each period must equal the inflow of new agents into the market. Therefore, given the trading probabilities p, q_1, and q_2, the steady-state measure of unmatched agents must satisfy the following equations:

$$\underbrace{(1-z_1)}_{\text{flexible worker entry}} = \underbrace{Wn_1m_1n_2p}_{\text{Trade with flexible firms}} + \underbrace{Wn_1m_1c_2q_2}_{\text{Trade with inflexible firms}} + \underbrace{Wn_1(1-m_1(n_2p+c_2q_2))\delta}_{\text{exogenous exit}}, \tag{1}$$

$$z_1=Wc_1m_1n_2q_1+Wc_1(1-m_1n_2q_1)\delta, \tag{2}$$

$$(1-z_2)l=\delta Fn_2+(1-\delta)Fn_2m_2(n_1p+c_1q_1), \tag{3}$$

$$z_2l=\delta Fc_2+(1-\delta)Fc_2m_2n_1q_2, \tag{4}$$

where $\delta = 1-e^{-\rho\tau}$ is the exogenously given exit rate.[11,12,13] For example, equation (1) is the steady-state equation for the flexible workers: the left-hand side has the total inflow of flexible workers coming into the market while the right-hand side has the total outflow of flexible workers due to successful trades plus the total outflow due to exogenous exits.

In what follows we will find it more convenient to work with the following steady-state equations which the steady-state frequency of inflexible types c and the market tightness parameter m must satisfy:

$$m = \frac{\delta}{\delta l + (1-\delta)(l-1)\pi}, \tag{5}$$

$$\frac{n_1}{c_1} = \frac{1-z_1}{z_1} \frac{\delta + (1-\delta)n_2 q_1}{\delta + (1-\delta)(n_2 p + c_2 q_2)}, \tag{6}$$

$$\frac{n_2}{c_2} = \frac{1-z_2}{z_2} \frac{\delta l + (1-\delta)((l-1)\pi + n_1 q_2)}{\delta l + (1-\delta)((l-1)\pi + n_1 p + c_1 q_1)}. \tag{7}$$

The following lemma asserts the existence and uniqueness of a steady-state distribution that satisfies equations (5), (6), and (7) for any given vector (δ, l, p, q, z). We describe how one can derive equations (5), (6), and (7) from equations (1)–(4) within the proof of Lemma 1 which is given in the Appendix. Intuitively, equations (6), and (7) assert that the relative likelihood of types is equal to the ratio of entry rates multiplied by the ratio of exit rates.

Lemma 1. *For any given vector* (δ, l, p, q, z)*, a unique steady-state distribution* (c, m) *that satisfies the steady-state equations* (5), (6), *and* (7) *exists. Moreover, the map from the set of vectors* (δ, l, p, q, z) *to the set of steady-state distributions* (c, m) *is continuous.*

Remark 1. *For any given vector* (δ, l, p, q, z)*, the steady-state distribution can be fully described by the market tightness parameter* m *and the steady-state frequencies* c_1 *and* c_2*. One can obtain the steady-state measure of workers* W *by simply dividing the inflow of workers, which is equal to one, by the total outflow of workers, which is equal to* $\delta + (1-\delta)\pi$*. We therefore omit the reference to the measure of workers,* W*, when describing a steady state.*

2.4. *Equilibrium*

We denote by $\Gamma(\Delta, c, v)$ the bargaining stage game where the time between offers is $\Delta \geq 0$, opting out is worth v_i to player i, and the initial belief that player i's opponent is inflexible, $\mu_i(h^0)$ is equal to c_i. In the bargaining stage game $\Gamma(\Delta, c, v)$, let $U_i(\Delta, \sigma, v|n)$ and $U_i(\Delta, \sigma, v|c)$ denote the payoff for a flexible player i conditional on facing a flexible or an inflexible opponent, respectively and let $U_i(\Delta, \sigma, v) = (1-c_j)U_i(\Delta, \sigma, v|n) + c_j U_i(\Delta, \sigma, v|c)$ denote the expected payoff for a flexible player i, if the agents use strategy profile σ.

11. The steady-state equations for the firms, equations (3) and (4), appear different from those for the workers, equations (1) and (2), because they have been rearranged to read more succinctly.

12. Note that in formulating the steady-state equations, we have ignored the fact that some trades may occur with delay and just focused on the probabilities with which agents eventually trade. This approach does not cause problems in the calculation of the steady-state distribution. This is because although some agents may trade with delay, the measure of agents who eventually trade must equal the measure of agents entering the steady-state population for the steady-state to be maintained.

13. Note that we implicitly assume that agents do not die (or leave the market due to the exogenous exit rate) while bargaining. This assumption greatly simplifies the steady-state calculation but has no impact on the results presented in the article.

$E(\Delta,\tau,\rho,l,z)$ denotes the search market. A *search equilibrium* s for $E(\Delta,\tau,\rho,l,z)$ is comprised of a strategy σ_k for each flexible player; a belief function μ_k for each player; steady-state frequencies of inflexible players $(c_1,c_2)\in[0,1]\times[0,1]$; and a market tightness parameter $m\in[0,1]$. More precisely, an equilibrium s satisfies the following conditions:

(i) The strategy profile σ and the belief profile μ comprises a *perfect Bayesian equilibrium* (PBE) in the bargaining stage-game $\Gamma(\Delta,c,v(s))$, where c_1 and c_2 are the equilibrium frequencies of inflexible players and $v_i(s)$ is the equilibrium value for a flexible player i.

(ii) The equilibrium values satisfy the following recursive equation for each i:

$$v_i(s)=e^{-(r_i+\rho)\tau}m_iU_i(\Delta,\sigma,v_i(s))+(1-m_i)e^{-(r_i+\rho)\tau}v_i(s). \tag{8}$$

In words, player i waits for τ units of time, remains in the market with probability $e^{-\rho\tau}$ before he is available to be paired, player i is paired with a bargaining partner with probability m_i, and remains unemployed in the market for another τ unit of time with probability $1-m_i$.

(iii) The market remains in steady state, *i.e.* the equilibrium frequencies c_1 and c_2, and the market tightness parameter m together satisfy equalities (5), (6), and (7) given the equilibrium strategy profile σ.

3. MARKET EQUILIBRIA

We say that a search equilibrium s of $E(0,\tau,\rho,l,z)$ is an *equilibrium with selective break ups* (SBU equilibrium) if at the bargaining stage, *a player opts-out only after he is finished conceding to the inflexible demand of his opponent.* In particular, an SBU equilibrium satisfies the following property:

Property 1. *For $i\in\{1,2\}$, if $\alpha_i(t)>0$ for some $t\geq 0$, then $F_i(t)=F_i(t')$ for any $t'>t$.*

In what follows our main focus is on SBU equilibria. We focus on this class of equilibria because they are analytically tractable and yet they generate dynamics sufficiently rich to allow for frequently observed bargaining behaviour such as delays and break-ups. We further justify our focus on SBU equilibria in Remark 2 given below. In the following theorem, we show that an SBU equilibrium always exists.

Theorem 1. *There is at least one SBU equilibrium of the market $E(0,\tau,\rho,l,z)$. All SBU equilibria have the following properties:*

(*a*) *Both players finish yielding concurrently at a common time T.*

(*b*) *If $T>0$, then after $t=0$, the players yield at constant hazard rates λ_1 and λ_2 where $\lambda_i:=\frac{r_j(1-\theta_i)}{(\theta_i+\theta_j-1)}$.*

(*c*) *At most one flexible player, i, (we call this player the "weak player") opts-out with positive probability at time T and never opts-out at any other time. Therefore, $\alpha_i(t)=0$ for $t<T$ and $\alpha_i(t)=\alpha_i(T)$ for $t\geq T$.*

(*d*) *Player j, who does not opt out, yields with an atom at $t=0$.*

(*e*) *Flexible players trade with certainty, i.e. $p=1$.*

An SBU equilibrium can deliver two types of bargaining dynamics. In the first one, which we call one-sided reputation dynamics, agreement is reached immediately, *i.e.* $T=0$. One of the players receives his inflexible demand right away, and if agreement is not reached, then he opts-out of the negotiation to look for a new match. His opponent, if flexible, yields to the inflexible demand of her opponent, with the correct anticipation that not yielding will result in a break-up, which is costly for her. This leads to the following definition.

Definition 1. *We say that an SBU equilibrium is an equilibrium with one-sided reputation for player i if at the bargaining stage, flexible players agree on an agreement immediately, player i receives θ_i from the flexible opponent, and player j receives $1-\theta_i$ by yielding immediately. There is no delay in reaching an agreement among the flexible types, i.e. $T=0$.*

The second type of SBU equilibrium dynamics, which we call two-sided reputation dynamics, exhibits a positive expected delay in reaching an agreement, *i.e.* $T>0$. One of the players, say player j, yields at the beginning with a probability that leaves player i with an outside option value that is equal to $1-\theta_j$. If there is no agreement at $t=0$, then the players yield to each other at constant hazard rates λ_1 and λ_2, until some common time T. At time T, player j's flexible type has finished yielding, and player i opts-out of the negotiations at T if the players fail to reach an agreement until then. This leads to the following definition.

Definition 2. *We say that an SBU equilibrium is an equilibrium with two-sided reputation building if there is a positive expected delay in reaching an agreement, i.e. $T>0$.*

We now relate the dynamics of SBU equilibria to the inflow rates of workers and firms, while focusing attention on frictionless search ($\tau=0$). We look at two cases: first, we focus on the case where the inflow of workers is less than the inflow of flexible firms, *i.e.* $l(1-z_2)<1$, and second we turn to the case where the inflow of workers is more than the inflow of firms, *i.e.* $l(1-z_2)>1$. We will show that if the inflow of firms and workers is symmetric, then the unique SBU equilibrium involves two-sided reputation building. In contrast, if the inflows of firms and workers are sufficiently asymmetric, then one-sided reputation building can also be sustained in an SBU equilibrium.

Remark 2. *We restrict our attention to SBU equilibria because this class of equilibria allows for a wide range of behaviour such as one-sided reputation building (as in Definition 1 or Myerson (1991)) or two-sided reputation building (as in Definition 2 or Abreu and Gul (2000)). Our focus on SBU equilibria restricts equilibrium behaviour only in the case where one of the two agents strictly prefers yielding to opting out while the other agent is exactly indifferent between yielding and opting out. In this case, focusing on SBU equilibria allows for a tractable analysis of the bargaining dynamics which involve delay and opting-out on the equilibrium path. We discuss other possible equilibria in section 3.5.*

3.1. *The case of $l(1-z_2)<1$*

In what follows, we provide a detailed description of an SBU equilibrium when $l(1-z_2)<1$. In order to simplify the exposition, we *suppose that the search cost $\tau=0$ and $l=1$, and hence, $m_2=1$ and $\delta=0$, i.e.* we focus on $E(0,0,\rho,1,z)$.[14] Also assume that $z_i<z^*:=\min\{\frac{\lambda_1}{\lambda_1+\lambda_2},\frac{\lambda_2}{\lambda_1+\lambda_2}\}$ for $i\in\{1,2\}$.

14. In Theorem 3 we show that if $l(1-z_2)<1$ and if τ and ρ are small enough, then all SBU equilibria have the feature that $T>0$, and hence all SBU equilibria share the qualitative features of the equilibrium $E(0,0,\rho,1,z)$ outlined in the discussion of this subsection.

As a preliminary step, we first describe the unique equilibrium of the bargaining game $\Gamma(0,c,0)$ in which opting out is strictly dominated by yielding to the inflexible demand for both players (or equivalently the game in which opting out is disallowed). This is the bargaining game analysed by Abreu and Gul (2000) and Compte and Jehiel (2002). Let

$$T_i := -\ln c_i/\lambda_i, \tag{9}$$

$$T := \min\{T_1, T_2\}, \tag{10}$$

$$b_i := c_i e^{\lambda_i T}. \tag{11}$$

In the unique equilibrium of this game, player i yields with a constant hazard rate λ_i and both players finish yielding at time T. The player i with the larger T_i is referred to as the "weak" player. The weak player i concedes with positive probability equal to $1-b_i$ at time zero. The strong player j concedes at time zero with probability zero, *i.e.* $1-b_j=0$. Note that equation (11) implies that $b_i = c_i c_j^{-\lambda_i/\lambda_j} \leq 1$ and $b_j = 1$ if $T_i \geq T_j$. After time zero, both flexible players are indifferent between conceding to an inflexible demand and continuing to resist. Therefore, the payoff of the strong flexible player j is equal to $(1-b_i)\theta_j + b_i(1-\theta_i)$ and the payoff of the weak flexible player i is equal to $(1-b_j)\theta_i + b_j(1-\theta_j) = 1-\theta_j$. These findings are summarized in Lemma B.1 in the Appendix.

Equilibrium play in an SBU equilibrium closely resembles equilibrium play in $\Gamma(0,c,0)$. However, there are two important differences: first, the weak player in an SBU equilibrium does not yield at time zero, instead he opts-out with positive probability once the two players have completed their yielding. See Figure 1 for a depiction. Second, the identity of the weak player in an SBU equilibrium is determined by the steady-state frequencies of inflexible agents in an auxiliary market. In this auxiliary market, the probabilities of facing inflexible opponents, *i.e.* $z_1/(1-z_2)$ and $z_2/(1-z_1)$, respectively, are set to equal the steady-state frequencies of inflexible types calculated under the assumption that flexible players always trade with their opponent (*i.e.* under the assumption that $p=q_1=q_2=1$).[15] More specifically, the weak player in $\Gamma(0,\hat{c},0)$, where $\hat{c}_i := \frac{z_i}{1-z_j}$ is also the weak player in the search market. Assume without loss of generality that player 1 is the weaker player in $\Gamma(0,\hat{c},0)$.

No atom at time zero: If $\tau=0$, then no player yields with an atom at the start of the bargaining game (*i.e.* at $t=0$). Because if player j yields with an atom, then her opponent i's equilibrium payoff at time zero would strictly exceed what she would get by yielding to an inflexible opponent. This, however, contradicts the fact that her opponent's outside option, v_i, is no more than $1-\theta_j$. Consequently, $F_1(t) = 1-e^{-\lambda_1 t}$ and $F_2(t) = 1-e^{-\lambda_2 t}$ for $t \leq T$ in an SBU equilibrium.

Finding T, q_2, c_1, and c_2: Because a flexible player 2 does not opt out, yields at rate λ_2, and finishes yielding at time T the following equation must be satisfied in an SBU equilibrium:

$$1-e^{-\lambda_2 T} = n_2 = 1-c_2. \tag{12}$$

Solving for T in equation (A.5) we obtain that $T=-\ln c_2/\lambda_2$. Thus, both players complete yielding by time $T=-\ln c_2/\lambda_2$. Since player 1 does not opt-out until T, the flexible players always trade, *i.e.* $p=1$, and the flexible player 2 always trades with an inflexible opponent, i.e, $q_1=1$. Also, because the total probability that player 1 yields by time T is equal to the total probability that player 1 trades with an inflexible opponent and because only the flexible player 1 trades with an

15. Note that the identity of the weak player, i.e. player who is weaker in the bargaining game $\Gamma(0,z_1/(1-z_2),z_2/(1-z_1)),0)$, only depends on the exogenously given variables z_1, z_2, θ_1, θ_2, r_1, and r_2.

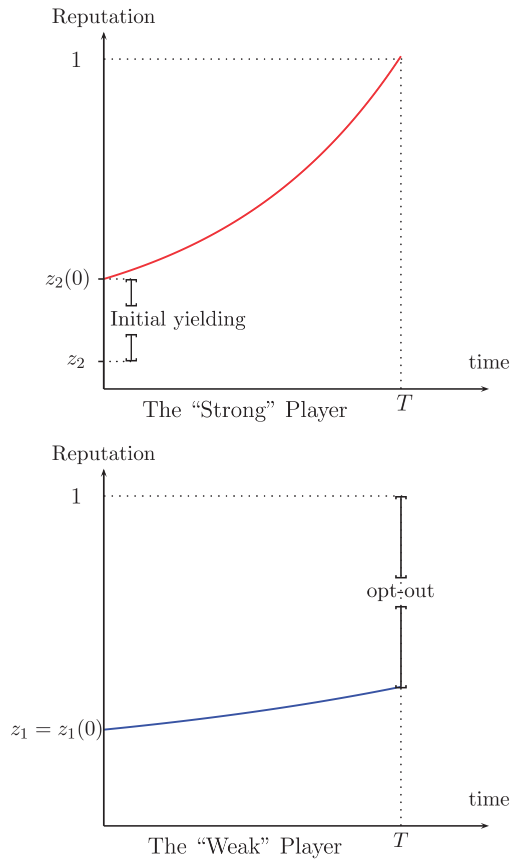

FIGURE 1

Player 2 yields with a hazard rate λ_2, whereas player 1 yields with a hazard rate λ_1 until time T. At $t=T$, conditional on nobody yielding by time T, player 1 opts-out. Player 1's immediate reputation level after $t=0$, $z_1(0)$, is equal to z_1. Player 2's immediate reputation level after $t=0$ (*i.e.* conditional on player 2 not yielding immediately), $z_2(0)$, is strictly greater than z_2

inflexible opponent we have that the following equation must be satisfied in an SBU equilibrium:

$$1-e^{-\lambda_1 T}=q_2(1-c_1). \tag{13}$$

Finally, the steady-state equations imply that the following two equations must also be satisfied in an SBU equilibrium:

$$c_1=\frac{z_1}{1-z_2} \tag{14}$$

$$c_2=\frac{z_2}{z_2+(1-z_1-z_2)q_2}. \tag{15}$$

Solution to 12, 13, 14, and 15: Note that, the previous step gives us four equations and four unknowns. This system of equations has a unique solution. Moreover, there is no solution to the similar set of equations that would prevail had we assumed that player 2 is the player who is opting out. Therefore, there is a unique SBU equilibrium of $E(0,0,\rho,1,z)$. We formally prove this in Lemma A.1 that can be found in the Appendix.

Remark 3. *If $\tau > 0$ but is sufficiently close to zero, then there is an SBU equilibrium that is similar to the the unique SBU equilibrium of the market when $\tau = 0$.*[16] *Moreover, if $l(1-z_2) < 1$ and if τ and ρ are positive but sufficiently small, then all SBU equilibria share the qualitative features of the equilibrium described in this subsection, involve two-sided reputation building and delays. We show these formally in Theorem 3.*

Remark 4. *If τ is sufficiently large, then the equilibrium of Abreu and Gul (2000), where neither player opts-out, is the unique equilibrium of this market. If τ is small, then this equilibrium can no longer be sustained. This is because the high payoff that the strong player receives in the bargaining game in the Abreu and Gul (2000) equilibrium and small τ together imply that this player's outside option exceeds conceding to an inflexible opponent. However, such outside options are not compatible with Abreu and Gul (2000)'s equilibrium as argued by Compte and Jehiel (2002).*

3.2. *The case of $l(1-z_2) > 1$*

Suppose now that $l(1-z_2) > 1$ and assume $\tau = 0$ and hence $\delta = 0$. In such a case, an SBU equilibrium where player 1 opts-out at $t = 0$ and player 2 yields at time $t = 0$ exists. In this equilibrium, the worker receives a payoff equal to θ_1 from his flexible opponent, and the firm receives a payoff equal to $1-\theta_1$. Agreement is reached immediately by the flexible players. The worker opts-out against the inflexible firm. Hence, in this SBU equilibrium, there is one-sided reputation for the worker because his outside option is greater than his payoff from yielding to his inflexible opponent. We discuss such equilibria in greater length in Theorem 3.

Remark 5. *If τ is sufficiently large, then the unique equilibrium of Abreu and Gul (2000) in which there is no outside option is an SBU equilibrium. However, when τ is small, then the SBU equilibrium with one-sided reputation for the workers always exists. Moreover, for a range of z_1 and z_2 values, there is also an SBU equilibrium with two-sided reputation building.*

3.3. *The limiting case of complete information*

We now turn to characterizing the limit outcomes as the entering measure of inflexible types, $z_1 + z_2$, approaches zero. Moreover, in order to single out *delay with in bargaining* as the source of inefficiency, we assume that $l = 1$, and hence $m_2 = 1$, and that $\tau = 0$, *i.e.* we continue analysing the unique SBU equilibrium of $E(0,0,\rho,1,z)$.

In Theorem 2 below, we show that along a sequence of markets with vanishing entry of inflexible types, the probability that player 1 trades with an inflexible type 2 goes to one and the steady-state frequency of inflexible types approaches zero. Since the search cost is zero, finding a bargaining partner is costless. Consequently, finding a flexible type as a bargaining partner is costless in such equilibria. However, even though the market is frictionless and

16. However, in such a case, there are two more SBU equilibria. In these equilibria, one of the players, say i, trades with an inflexible opponent with a very small probability, and opts-out very quickly. When τ is small, so is the exit rate δ. Therefore, this trading pattern makes the steady-state fraction of inflexible opponents very close to one. There is an initial gift from the flexible opponent, however since the probability of finding a flexible opponent is small, the outside option of such a player is equal to $1-\theta_j$ in such an equilibrium.

(asymptotically) free of inflexible types, inefficiency in the bargaining game remains substantial (*i.e.* $v_1 = 1-\theta_2$ and $v_2 = 1-\theta_1$). Hence, Theorem 2 leads us to conclude that access to the market exacerbates bargaining inefficiencies caused by inflexible types, instead of forcing outcomes closer to efficiency, even as we approach the frictionless model without inflexible types.

Theorem 2. *Let* $\{E(0,0,\rho,1,z^k)\}_{k\in\{1,2,...\}}$ *be a sequence of search markets, let* s^k *denote an SBU equilibrium of* $E(0,0,\rho,1,z^k)$*, let* c^k *denote the steady-state frequency of inflexible types,* q_1^k *and* q_2^k *denote the probability with which the flexible types trade with the inflexible types in this equilibrium. If the entering measure of inflexible types converges to zero, i.e. if* $\lim_k z_1^k + z_2^k = 0$*, then* $\lim_k q_1^k = \lim_k q_2^k = 1$ *and the steady-state frequency of inflexible types converges to zero, i.e.* $\lim_k c_1^k = \lim_k c_2^k = 0$.

Theorem 2 stands in sharp contrast to the previous literature. In particular, Abreu and Gul (2000) show that one of the two players is asymptotically strictly stronger and if player i is asymptotically strictly stronger, then the two players trade immediately without any delay, the equilibrium payoff of the stronger side is θ_i and the equilibrium payoff of the weaker player is $1-\theta_i$, at the limit.[17] Hence, inefficiency disappears but incomplete information still has an impact on the division of the surplus, at the limit. In contrast, inefficiency and delay remain substantial in an SBU equilibrium even at the limit. As Theorem 2 argues, this inefficiency is not an artifact of the fact that the inflexible type probabilities stay bounded above zero. On the contrary, the inflexible type probabilities also converge to zero as the inflow of inflexible types approach zero.

The main difference between an SBU equilibrium and the equilibrium in Abreu and Gul (2000) is the time at which the atom appears. In the war of attrition without outside options, the weaker player's equilibrium strategy has an atom at the beginning. As the inflexible type probabilities approach zero, the duration of the war of attrition goes to infinity. If the hazard rates are different, then without any initial atom, the weaker player's reputation stays close to zero at the time her opponent finishes yielding (see Figure 2). The implication of this is that the probability of the atom should be sufficiently close to one in order for both players' reputations to reach one at the same time. This results in efficiency because the expected delay goes to zero as the inflexible type probabilities approach zero.

In the SBU equilibrium, if the inflexible type probabilities approach zero, then the duration of the war of attrition approaches infinity. The reputation of the weaker player is close to zero at time her opponent finishes yielding. Hence, the weaker player opts-out with a probability close to one at the end of the war of attrition. Opt-out occurs, however, only if neither player has yielded to their opponent by the end of the war of attrition. Since the duration of the war of attrition is long, and the players yield with a constant hazard rate, the probability that players yield to each other approaches one. Hence, opt-out is realized conditional on an event whose probability approaches zero. In other words, from an ex-ante perspective (*i.e.* the beginning of the game) the probability of opt-out is very small. However, conditional on neither party yielding to each other, this probability is close to one. Therefore, both flexible types trade with their inflexible type opponent with a probability that approaches one.

Therefore, the key reason for our inefficiency results can be summarized as follows: if the atom is at the end of the war of attrition (as in our model), then it becomes increasingly unlikely that the game continues until the time where the player opts-out with an atom. In contrast, if the

17. In particular, if $\lambda_1 < \lambda_2$ and if the inflexible type probabilities approach to zero at the same rate, then player 2 is the stronger player, asymptotically. If the inflexible type probabilities converge at different rates, then generically one of the two players is strictly stronger, asymptotically.

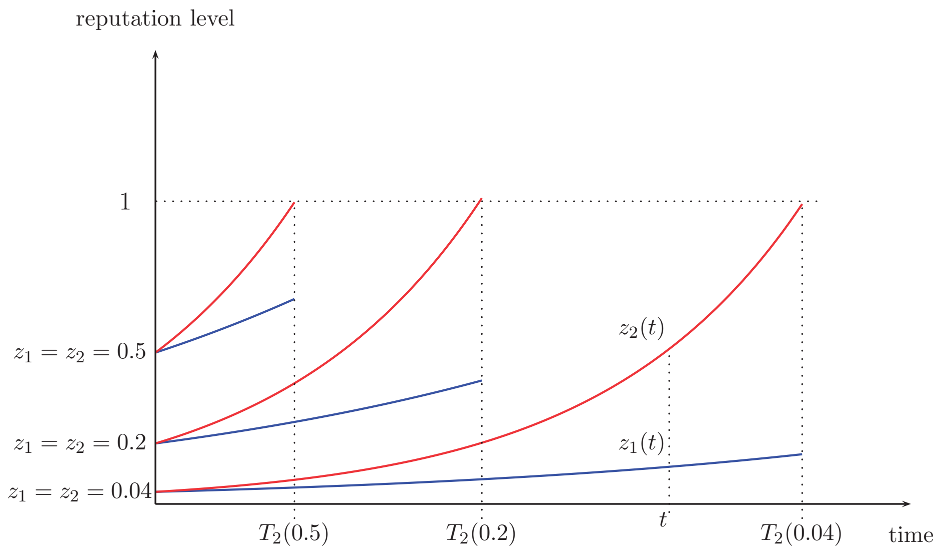

FIGURE 2

This figure depicts the evolution of the reputations of both players along the path where none of the players have yielded. Note that if $\lambda_1 < \lambda_2$, and if z_1 and z_2 are very close to zero, then $z_1(T_2)$ stays close to zero

atom is at the beginning of the war of attrition (as in Abreu and Gul (2000)), then it is reached immediately.

3.4. *Properties of SBU equilibria*

We now discuss more general properties of SBU equilibria when the search frictions are small, *i.e.* when ρ and τ are close to zero, and when there is no friction within the bargaining stage, *i.e.* $\Delta = 0$.

Theorem 3.

(*i*) *Fix* $z_1, z_2 > 0$ *and suppose that* $l(1-z_2) < 1$. *There is a* $\bar{\rho} > 0$ *and* $\bar{\tau} > 0$, *such that if* $\rho < \bar{\rho}$ *and if* $\tau < \bar{\tau}$, *then the following are true for every SBU equilibrium* s *of* $E(0,\tau,\rho,l,z)$:

- (*a*) $v_i \leq 1-\theta_{-i}$ *for* $i \in \{1,2\}$ *where one of the two inequalities holds strictly.*
- (*b*) *Both inflexible workers and inflexible firms consummate a match with a strictly positive probability.*
- (*c*) *There is two-sided reputation building.*
- (*d*) *There are delays in reaching an agreement in the bargaining stage.*

(*ii*) *Fix* $z_1, z_2 > 0$ *and suppose that* $l(1-z_2) > 1$. *There is a* $\bar{\rho} > 0$ *and* $\bar{\tau} > 0$, *such that if* $\rho < \bar{\rho}$ *and if* $\tau < \bar{\tau}$, *then there is an SBU equilibrium* s *of* $E(0,\tau,\rho,l,z)$ *in which the following are true:*

- (*a*) $v_1 > 1-\theta_2$.
- (*b*) *Inflexible firms never reach an agreement.*

(c) Workers have one-sided reputation.
(d) The bargaining stage is efficient and there is immediate agreement.

The theorem first considers the case where the inflow rate of the flexible firms is less than the inflow rate of all workers. In this case, if the search frictions τ and ρ are small, then all SBU equilibria involve two-sided reputation building. As evident from the low outside option values, these equilibria are always inefficient. Moreover, in such equilibria the expected delay while bargaining is positive because $T > 0$. The intuition for why the outside option values are low is as follows: if the outside option of one side, say the worker, is higher than $1-\theta_2$, then a flexible worker would never consummate a match with an inflexible firm. However, if the death rate ρ is sufficiently small, then inflexible firms who never consummate a successful match would be much more frequently encountered than flexible firms. In turn, a worker would have to spend a lot of time to locate a flexible firm. But then the cost of searching for such a firm would push the worker's outside option below $1-\theta_2$, yielding a contradiction.

Alternatively, if the inflow rate of the flexible firms is more than the inflow rate of all workers, then when the search frictions are small, there exists an SBU equilibrium which has one-sided reputation for the workers. In this equilibrium, the flexible types trade immediately, and the firms agree to the demand of the inflexible worker. Since the inflow rate of the flexible firms is high, they are not scarce relative to the inflexible firms in the market. Hence, as the search frictions disappear, the equilibrium payoffs of the workers converge to θ_1 and the bargaining stage payoffs of the firms is equal to $1-\theta_1$. The intuition for this result is as follows: in a candidate search equilibrium where there is one-sided reputation building for the workers, the inflexible firms never consummate a successful match. Flexible firms, in contrast, always trade if they are matched with a worker. However, the inflow rate of flexible firms is more than the inflow rate of workers, therefore, every period there will be a positive fraction of flexible firms available to match with a partner. Hence, in steady state the fraction of flexible firms in the market is bounded away from zero, even when the search frictions disappear. Therefore, if τ is small, then a flexible worker can find a flexible firm without encountering search frictions.

Remark 6. *Note that if the search cost τ is sufficiently large, then players' outside options are smaller than the inflexible opponents' demands. In this case, there is a unique equilibrium where bargaining follows the Abreu and Gul (2000) equilibrium. Alternatively, if τ is small, then there are potentially multiple SBU equilibria. If $l(1-z_2) > 1$, then there is an SBU equilibrium with one-sided reputation. This equilibrium is efficient and exhibits immediate agreement.*[18] *On the other hand, if $l(1-z_2) < 1$, then all SBU equilibria involve two-sided reputation building and hence all SBU equilibria entail delays as asserted in Theorem 3.*

3.5. *Other equilibria*

In this article, we primarily focused on a class of equilibria termed SBU equilibria where a player never yields to an opponent after any time t if that player also opts-out with positive probability before time t. There are also other equilibria of our model.

18. In this case, multiplicity may arise. In particular, for some parameters z_1, z_2, there is also a search equilibrium in which there is two-sided reputation building in the bargaining stage. More precisely, for every $z_2 > 0$, there exists a $\bar{z}_1 > 0$ such that if $z_1 < \bar{z}_1$, then an additional two-sided equilibrium appears. In this equilibrium $v_1 = 1-\theta_2$, the workers trade with their inflexible opponent with a positive probability, and opt-out with a positive probability at the end of the war of attrition.

Suppose that equilibrium values are such that $v_i = 1-\theta_j$ for some i. In this case, there are a continuum of other equilibria where player i opts-out and yields concurrently with an atom at some time t before the war of attrition is over. Such equilibria can be sustained, if the relative probabilities with which player i yields and opts-out at t leaves player j indifferent between yielding and waiting further. These equilibria are qualitatively similar to SBU equilibrium with two-sided reputation building but far less tractable.

Instead, if equilibrium values are such that $v_1 \neq 1-\theta_2$ and $v_2 \neq 1-\theta_1$, then equilibrium behaviour is uniquely pinned down. Note however that such a configuration of outside options cannot arise if $l(1-z_2)<1$ and if the search frictions τ and ρ are sufficiently small, as we demonstrated in Theorem 3. Therefore, if $l(1-z_2)<1$, then all equilibria are qualitatively similar to SBU equilibria. Alternatively, if $l(1-z_2)>1$, then there is an equilibrium with one-sided reputation building where $v_1 > 1-\theta_2$ and $v_2 < 1-\theta_1$ for sufficiently small τ and ρ. In this case, however, equilibria are not necessarily unique even for sufficiently small τ and ρ as there is also an SBU equilibrium with two-sided reputation building for a certain range of parameter values. In this SBU equilibrium, the outside option of player 2 is low because he is on the long side of the market, and the outside option of player 1 is equal to $1-\theta_2$.[19]

There is yet another source of multiplicity if the bargaining stage is modelled as a discrete-time bargaining game instead of a war of attrition. We focus on such games in the Appendix. The formulation of the bargaining stage as a war of attrition does not allow for the possibility that the flexible types may reveal their types and then trade at terms which are different from the inflexible type's demand. So a natural question to ask in the discrete time model is whether there exists a search equilibrium in which only the flexible types reach an agreement. In Lemma B.6 we show that such an equilibrium does not exist. In particular, if the search time τ and exit rate ρ are sufficiently small, then there is no search equilibrium in which only flexible types trade and both flexible types' outside options exceed the inflexible type demands. The intuition for this result is as follows: if the inflexible types never consummate a successful match, and if the exit rate is small, then the inflexible types are plentiful in the market. Thus it becomes difficult for one side of the market to find a flexible opponent. In this candidate equilibrium, a flexible type goes through many unsuccessful matches before reaching an agreement. Even though the search cost is small, the impact of the low exit rate dominates the ease of finding a new match thus resulting in low outside options for both flexible types and contradicting that the flexible types opt-out against their inflexible opponents.

4. CONCLUSION

This article develops and analyses a stylized model of a two-sided search market. The model captures three prominent features of the markets for labour and housing: (i) market conditions determine outside options; (ii) private information about past commitments or one's degree of flexibility tempts agents to adopt inflexible bargaining postures; (iii) negotiations occasionally break down. We show that there always exists an equilibrium where bargaining parties either immediately settle on an agreement or feign inflexibility to build a reputation and opt-out only after negotiating without an agreement for a payoff relevant length of time. When the search frictions are small, (i) if the inflow of workers is not less than the inflow of flexible firms, then all such equilibria exhibit low outside options, delays in reaching an agreement, and occasional bargaining break-downs; (ii) if the inflow of workers is less than the inflow of flexible firms, then

19. Such SBU equilibria do not always exist because the set of equations that need to be satisfied in an SBU equilibrium may fail to have a solution when player 1 is the player who opts-out of the bargaining stage.

an equilibrium in which the bargaining phase ends with an immediate agreement or break-up exists. These results help us understand the impact of uncertainty about bargaining postures on the prevalence of bargaining delays and negotiation break downs when such negotiations take place in a market context.

If bargaining takes place in isolation from markets, as shown by previous research, a small amount of uncertainty may result in only short expected delay times in bargaining. However, in our model, when the parties have access to the market to find a new match, arbitrarily long delays arise in equilibrium, as long as there is some uncertainty, and as long as there is a relatively symmetric inflow of workers and firms to the market. Asymmetric inflows, however restore efficiency, and results in immediate agreement in negotiations.

In this article, we assumed that there is a constant inflow of workers and firms into the market and further assumed a steady state to generate an endogenous distribution of types. An alternative modelling choice would be to assume cloning where each player is replaced by an identical clone when he exits the market (Rubinstein and Wolinsky, 1985). In such a formulation, an equilibrium where only flexible types consummate a successful match can be sustained if the fraction of inflexible types is not too high. However, such a model does not capture the intuitive idea that the relative frequency of types in the market should be determined by frequency with which each type trades.

APPENDIX

A. PROOFS OF LEMMA 1, LEMMA A.1, THEOREM 1, THEOREM 2 AND THEOREM 3

Proof of Lemma 1. First we show how we derive the steady-state equations (5), (6), and (7): adding equation (1) to (2), adding equation (3) to (4), and noticing $F=W/m$ we obtain the following aggregate steady-state equation for the workers and the firms:

$$1=\delta W+(1-\delta)Wm_1\pi, \tag{A.1}$$

$$l=\delta(W/m)+(1-\delta)(W/m)m_2\pi. \tag{A.2}$$

Note that $m<1$ if $l>1$ and $m=1$ if $l=1$. This is because the total measure of workers who form a match must equal the total measure of firms who form a match, *i.e.* $Wm_1\pi=Fm_2\pi=\frac{W}{m}m_2\pi$. Therefore, subtracting equation (A.1) from (A.2) we find that $\frac{W}{m}-W=(l-1)/\delta=F-W$. Hence, $F=W$, *i.e.* $m=1$, if $l=1$, and $F>W$, *i.e.* $m<1$, if $l>1$. Since we have $l\geq 1$ by assumption, we get that $m\leq 1$, $m_1=1$ and $m_2=m$.

Substituting $m_1=1$ and $m_2=m$, then dividing equation (A.1) by equation (A.2) we obtain $\frac{1}{l}=\frac{\delta m+(1-\delta)m\pi}{\delta+(1-\delta)m\pi}$ and rearranging we get $m=\frac{\delta}{\delta l+(1-\delta)(l-1)\pi}$, *i.e.* equation (5).

Substituting $m_1=1$ into equations (1) and (2) then dividing (1) by (2) we get $\frac{1-z_1}{z_1}=\frac{Wn_1(\delta+(1-\delta)(n_2p+c_2q_2))}{Wc_1(\delta+(1-\delta)n_2q_1)}$. We obtain equation (6) by rearranging this expression.

Substituting $m_2=m$ into equations (3) and (4) then dividing (3) by (4) we get:

$$\frac{1-z_2}{z_2}=\frac{Fn_2(\delta/m+(1-\delta)(n_1p+c_1q_1))}{Fc_2(\delta/m+(1-\delta)n_1q_2)}.$$

Equation (5) implies that $\delta/m=\delta l+(1-\delta)(l-1)\pi$. We obtain equation (7) by substituting $\delta l+(1-\delta)(l-1)\pi$ for δ/m and rearranging.

The proof of existence and uniqueness follows from a slight modification of the existence and uniqueness proof of Noldeke and Tröger (2009) by setting $\rho=1$ in their proof. In the market we analyse, there are only two types for each side, hence a straightforward application of Kakutani's fixed point theorem yields existence. Upper-hemicontinuity of the fixed-point correspondence is standard. Showing uniqueness follows from defining a suitable contraction mapping, and applying the technique of Noldeke and Tröger (2009). Alternatively, the argument in Lauermann (Forthcoming) applies in our setup. ‖

Lemma A.1. *If* $z_1,z_2<z^*=\min\{\frac{\lambda_1}{\lambda_2+\lambda_1},\frac{\lambda_2}{\lambda_2+\lambda_1}\}$, *then there is a unique SBU equilibrium of* $E(0,0,\rho,1,z)$.

Proof Note that, because we assume that $z_i < z^* = \min\{\frac{\lambda_1}{\lambda_2+\lambda_1}, \frac{\lambda_2}{\lambda_2+\lambda_1}\}$, $z_1 + z_2 < 1$. Below we will show that there exists a unique $q_2 \in (0,1]$ such that equations (12) and (13) and steady-state equations (14) and (15) are satisfied. Also, we show that there exists a $q_2 \in (0,1]$ such that equations (12)–(15) are satisfied, only if player 1 is the weaker player in $\Gamma(0,\hat{c},0)$ where $\hat{c} = (z_1/(1-z_2), z_2/(1-z_1))$. We also show that the case when $q_2 = 0$ cannot be an equilibrium. Consequently, there is a unique SBU equilibrium, if player 1 is assumed to be the weaker player in $\Gamma(0,\hat{c},0)$. Since either player 1 or player 2 is the weaker player in $\Gamma(0,\hat{c},0)$, our argument establishes that there is a unique SBU equilibrium.

Suppose that player 1 is weaker in $\Gamma(0,\hat{c},0)$ and define $b_2(q_2)$ using the following equation:

$$1 - b_2(q_2)e^{-\lambda_2 T(q_2)} = 1 - c_2(q_2).$$

Substituting in for $T(q_2)$ and $c_2(q_2)$ gives the following expression for $b_2(q_2)$

$$b_2(q_2) = \frac{z_2}{z_2 + q_2(1-z_1-z_2)}\left(\frac{1-z_2}{(1-z_2) - q_2(1-z_1-z_2)}\right)^{\frac{\lambda_2}{\lambda_1}}.$$

A necessary condition for equilibrium is $b_2(q_2) = 1$ since player 2 does not yield at time zero. We show that there exists a unique $q_2 > 0$ such that $b_2(q_2) = 1$. Notice that $b_2(0) = 1$ as well. However, there cannot be an equilibrium in which $q_2 = 0$. This is because, $q_2 = 0$ implies that $c_2 = 1$ and thus $n_2 = 0$. This implies that player 1 neither trades with inflexible type 2 ($q_2 c_2 = 0$) nor player 2 ($pn_2 = 0$). Hence, $v_1 = 0$. But this cannot be an equilibrium because if $v_1 < 1 - \theta_2$, then opting out is strictly dominated by yielding, and hence contradicting that $q_2 = 0$. In the following development, we will show that there is a unique $q_2 > 0$ for which $b_2(q_2) = 1$.

We rewrite the expression for $b_2(q_2)$ to obtain the following:

$$b_2(q_2) = f(k(q_2))^{1/\lambda_1}$$

where $k(q_2) = z_2 + q_2(1-z_1-z_2)$ and

$$f(k) = z_2^{\lambda_1}(1-z_2)^{\lambda_2} k^{-\lambda_1}(1-k)^{-\lambda_2}.$$

If $q_2 = 1$, then $b_2(1) \geq 1$ (since player 1 is weaker in $\Gamma(0,\hat{c})$). The function $f(k)$ is strictly convex and minimized at $k = \frac{\lambda_1}{\lambda_2+\lambda_1} \in (0,1)$. Let

$$p^* = \frac{\frac{\lambda_1}{\lambda_1+\lambda_2} - z_2}{1-z_1-z_2} \tag{A.3}$$

and observe that $k(p^*) = \frac{\lambda_1}{\lambda_2+\lambda_1}$. Our assumption that $z_i < z^*$ implies that $p^* \in (0,1)$. Also, notice $b_2(p^*) < b_2(0) = 1 \leq b_2(1)$. Consequently, the convexity of f implies that f is decreasing for $q_2 \leq p^*$ and increasing for $q_2 \geq p^*$. This implies that $f(k(q_2)) < 1$ for all $q_2 \leq p^*$ and there exists a unique $q_2 \in (p^*, 1]$, such that $b_2(q_2) = 1$. Note that if player 2 is strictly weaker in $\Gamma(0,\hat{c},0)$, then doing the same analysis in the mirror problem where player 2 is supposed to opt-out, we would obtain that $b_1(1) < 1$ and so there does not exist a $q_1 \in (0,1]$ such that $b_1(q_1) = 1$. Hence the equilibrium is unique. ‖

Proof of Theorem 2. First, without loss of generality, we pick a subsequence where player 1 is the weaker player in $\hat{\Gamma}(0,\hat{c}^k)$ for all k. Moreover, because $\lim_{k\to\infty} z_i^k = 0$, there is an integer K such that for every $k > K$, $z_i < z^*$ for $i \in \{1,2\}$. In the following, we renumber the sequence so that its first element is the $(K+1)^{st}$ element of the original sequence. The argument for the proof of Lemma A.1 implies that $q_2^k \geq p*^k$ for all k, where $p*^k$ is defined in equation (A.3) using z_1^k and z_2^k. Notice $\lim_k p*^k = \lambda_1/(\lambda_1+\lambda_2)$. Consequently, $\lim_k q_2^k \geq \lambda_1/(\lambda_1+\lambda_2) > 0$. Again the argument for Lemma A.1 delivers:

$$b_2(q_2^k) = \frac{z_2^k}{z_2^k + q_2^k(1-z_1^k-z_2^k)}\left(\frac{1-z_2^k}{(1-z_2^k) - q_2^k(1-z_1^k-z_2^k)}\right)^{\frac{\lambda_2}{\lambda_1}} = 1. \tag{A.4}$$

for all k. Consequently, $\lim_k b_2(q_2^k) = 1$. If $\lim_k b_2^k(q_2^k) = 1$, then equation (A.4) implies that either $\lim_k q_2^k = 0$ or $\lim_k q_2^k = 1$. However, $\lim_k q_2^k \geq \lambda_1/(\lambda_1+\lambda_2) > 0$ implies that $\lim_k q_2^k = 1$. Also, $\lim_k q_2^k = 1$ and equations (14) and (15) imply that $\lim_k c_1^k = \lim_k c_2^k = 0$. ‖

Proof of Theorem 1. The proof proceeds as follows. We first show that an SBU equilibrium that lasts for a strictly positive time (*i.e.* $T > 0$) has to be of the following form:

1. The players yield at constant rates λ_1 and λ_2 where $\lambda_i \equiv \frac{r_j(1-\theta_i)}{(\theta_i+\theta_j-1)}$.
2. Both players finish yielding concurrently at a common time T.
3. At most one player, say player i opts-out with positive probability at time T and never opts-out at any other time. Therefore, $\alpha_i(t) = 0$ for $t < T$ and $\alpha_i(t) = \alpha_i(T)$ for $t \geq T$.
4. If $\alpha_i(T) > 0$, then $v_i = 1 - \theta_j$, and $v_j < 1 - \theta_i$. Player j yields with an atom at $t = 0$.
5. The flexible players trade with certainty, *i.e.* $p = 1$.

If $\alpha_i(t)=\alpha_j(t)=0$ for every $t\geq 0$, then the equilibrium of the bargaining game is uniquely determined by Abreu and Gul (2000)'s analysis, and so the equilibrium properties claimed above are satisfied.

Suppose, without loss of generality, that $\alpha_1(t)>0$ for some $t>0$. Let $T^*=\inf\{t:\alpha_1(t)>0\}$. T^* denotes the first point in time where player 1 opts-out. Definition of the SBU equilibrium requires that player 1 does not yield after time T^*. Therefore, player 2's continuation payoff at any time $t>T^*$ is at most $1-\theta_1$, so she has no incentive to delay yielding or opting out beyond T^* and thus must complete yielding by time T^*. Player 1's continuation payoff at any time $t>T^*$ is at most $1-\theta_2$ because player 2 completes her yielding by time T^*. Consequently, player 1 has no incentive to delay opting out or yielding beyond time T^* and thus must complete yielding and opting out by time T^*.

Since $T>0$, $v_1\leq 1-\theta_2$ and $v_2\leq 1-\theta_1$. This is because, otherwise, if $v_i>1-\theta_j$, then negotiations would not last for a positive time, and either player j would yield at time 0 or player i would opt-out if not yielded to. Also, because player 1 opts-out with positive probability, where yielding was feasible, $v_1\geq 1-\theta_2$. Hence, $v_1=1-\theta_2$.

An argument that follows closely Proposition 1 in Abreu and Gul (2000) implies that the two agents concede at constant hazard rates λ_1 and λ_2, $v_1=1-\theta_2$, $v_2<1-\theta_1$ and $\alpha_2(t)=0$ for every $t\geq 0$.[20] Also, the previous discussion implies that both agents complete yielding by time T^*. Consequently, in any SBU equilibrium both players concede at constant hazard rates λ_1 and λ_2 and complete yielding by some common time $T<\infty$. Since $v_1=1-\theta_2$, $F_1(t)=1-e^{-\lambda_1 t}$ and $F_2(t)=1-be^{-\lambda_2 t}$ for $t\leq T$ in an SBU equilibrium, it has to be that $(1-b)\theta_1+b(1-\theta_2)=(1-\theta_2)(e^{\tau(\rho+r_1)})$, *i.e.* player j yields with an atom at $t=0$. Because player j finishes yielding by T, flexible types trade with each other with certainty. We have thus completed the proof for the properties of SBU equilibria asserted above.

If $T=0$ in an SBU equilibrium, then $v_i\geq 1-\theta_j$ and $v_j<1-\theta_i$ for some $i\in\{1,2\}$. On the way to a contradiction, first suppose that $v_i<1-\theta_j$ for $i\in\{1,2\}$. Then opting out would be dominated by yielding, and hence Abreu and Gul (2000)'s analysis would yield trade probabilities $p=q_1=q_2=1$. However, then steady-state equations then deliver $c_1,c_2>0$. But this contradicts that $T=0$. Again on the way to a contradiction, suppose that $v_i\geq 1-\theta_j$ for $i\in\{1,2\}$. If $F_i(0)>0$, then $F_j(0)=0$, because player j can opt-out at $t=0$ instead of yielding, and conditional on i yielding at $t=0$, strictly increases his payoff, and conditional on i opting out at $t=0$, does not lower his payoff, since $v_j\geq 1-\theta_i$. But then, if $F_j(0)=0$, and if $T=0$, then $v_i\leq e^{-(r_i+\rho)\tau}(1-\theta_j)<1-\theta_j$, which is a contradiction to $v_i\geq 1-\theta_j$.

Now we will show that an SBU equilibrium always exists.

Because a flexible player 2 does not opt-out, yields at rate λ_2, and finishes yielding at time T we have that the following equation must be satisfied in such an SBU equilibrium:

$$1-be^{-\lambda_2 T}=n_2=1-c_2. \tag{A.5}$$

Solving for T in equation (A.5) we obtain that $T=\frac{-\ln(c_2/b)}{\lambda_2}$. Note that both players complete yielding by time T. Therefore, the flexible players always trade, *i.e.* $p=1$, and the flexible player 2 always trades with an inflexible opponent, *i.e.* $q_1=1$. Also, because the total probability that player 1 yields by time T is equal to the total probability that player 1 trades with an inflexible opponent and because only the flexible player 1 trades with an inflexible opponent we have that the following equation must be satisfied in an SBU equilibrium:

$$1-e^{-\lambda_1 T}=q_2(1-c_1). \tag{A.6}$$

Finally, the steady-state equations imply that the following two equations must also be satisfied in an SBU equilibrium:

$$\pi=n_2+n_1c_2q_2, \tag{A.7}$$

$$\frac{n_1}{c_1}=\frac{1-z_1}{z_1}\frac{\delta+(1-\delta)n_2}{\delta+(1-\delta)(n_2+c_2q_2)}, \tag{A.8}$$

$$\frac{n_2}{c_2}=\frac{1-z_2}{z_2}\frac{\delta l+(1-\delta)((l-1)\pi+n_1q_2)}{\delta l+(1-\delta)((l-1)\pi+1)}. \tag{A.9}$$

Let $\Gamma_1:[0,1]\rightarrow\mathbb{R}$ be a function defined as follows: for any given $q_2\in[0,1]$, let $c(q_2)=(c_1(q_2),c_2(q_2))\in[0,1]^2$ be the unique vector of inflexible type frequencies that satisfy the above steady-state equations. Then, using q_2 and $c_1(q_2)$ in equation (A.6) to solve for T delivers a unique solution, called $T(q_2)$. Using $T(q_2)$, and $c_2(q_2)$ in equation (A.5) to solve for b, we obtain a unique solution, $b(q_2)\geq 0$. Now,

$$\Lambda_1(q_2):=((1-b(q_2))\theta_1+b(q_2)(1-\theta_2))e^{-(r_1+\rho)\tau}. \tag{A.10}$$

20. This is very standard by now, and hence we omit the details of the argument. The main difference is to rule out the case in which a player may opt-out and yield with atoms at time T, which follows because when a player yields and opponent opts-out, the player get payoffs from trading, and not opt-out.

Similarly, let us now define the mirror problem where player 2 is the player who opts-out at the bargaining stage. Then the equations that correspond to equations (A.5) – (A.9) become as follows:

$$1-e^{-\lambda_2 T}=q_1(1-c_2), \tag{A.11}$$

$$1-be^{-\lambda_1 T}=1-c_1, \tag{A.12}$$

$$m=\frac{\delta}{\delta l+(1-\delta)(l-1)\pi}, \tag{A.13}$$

$$\pi=n_1+n_2c_1q_1, \tag{A.14}$$

$$\frac{n_1}{c_1}=\frac{1-z_1}{z_1}(\delta+(1-\delta)n_2q_1), \tag{A.15}$$

$$\frac{n_2}{c_2}=\frac{1-z_2}{z_2}\frac{\delta l+(1-\delta)((l-1)\pi+n_1)}{\delta l+(1-\delta)((l-1)\pi+n_1+c_1q_1)}. \tag{A.16}$$

Similarly, let $\Lambda_2:[0,1]\to\mathbb{R}$ be defined similar to Λ_1, except that

$$\Lambda_2(q_1):=((1-b(q_1))\theta_2+b(q_1)(1-\theta_1))\frac{m}{e^{(r_2+\rho)\tau}-1+m}, \tag{A.17}$$

That is, there is an added possibility that player 2 may fail to be matched when he goes back to the pool of unmatched agents.

If $\Lambda_i(0)\geq 1-\theta_j$, then there is an SBU equilibrium where player i trades with inflexible opponents with probability zero, and enjoys a one-sided reputation payoff against the flexible opponent who yields right away. So now assume $\Lambda_i(0)<1-\theta_j$ for $i\in\{1,2\}$. Now consider $\Lambda_i(1)$. If $\Lambda_i(1)\leq 1-\theta_j$ for $i\in\{1,2\}$, then Abreu and Gul (2000)'s analysis applies in this setup, since the outside options are not more than yielding to an inflexible opponent, had the players not opted out against their opponents. The only remaining case is if $\Lambda_i(1)>1-\theta_j$ for some $i\in\{1,2\}$. In this case, since $\Lambda_i(0)<1-\theta_j$, and since $\Lambda_i(\cdot)$ is a continuous function, by the intermediate value theorem, there is a $q_j\in(0,1)$ such that $\Lambda_i(q_j)=1-\theta_j$. It is straightforward to check that this constitutes an SBU equilibrium. ‖

Proof of Theorem 3. Fix a z_1,z_2 and assume that $l(1-z_2)<1$. Suppose that $v_i>1-\theta_j$ for some $i\in\{1,2\}$. Then for all PBE of $\Gamma(0,c,v)$, $q_j=0$, and $p=1$.

Case 1: Suppose that $v_1>1-\theta_2$. Then $v_2<1-\theta_1$, because in equilibrium, player 1 opts-out immediately unless player 2 yields to him immediately. Therefore, the highest possible payoff at the bargaining stage for player 2 is $1-\theta_1$, and because search is costly, $v_2<1-\theta_1$.

Since $v_2<1-\theta_1$, the equilibrium of $\Gamma(0,c,v)$ is unique and $p=1,q_1=1,q_2=0$. Therefore, the steady-state equations are as below:

$$n_1 = \frac{(1-z_1)(\delta+(1-\delta)n_2)}{\delta+(1-\delta)n_2}=1-z_1,$$

$$n_2 = \frac{(1-z_2)(\delta l+(1-\delta)(l-1)n_2)}{\delta l+(1-\delta)(l-1)n_2+z_2(1-\delta)}.$$

Notice that, $v_1\leq e^{-(r_1+\rho)\tau}(n_2+c_2v_1)$, because $p=1,q_2=0$ and $1>1-\theta_2$. We remind the reader that $\delta=1-e^{-\rho\tau}$.

We will now show that, n_2, viewed as a function δ (or τ, at this point we are keeping ρ fixed) has the property that $\lim_{\delta\to 0}\frac{n_2(\delta)}{\delta}=\frac{(1-z_2)l}{z_2-(1-z_2)(l-1)}$. To see this, on the way to a contradiction, first suppose that there is a sequence $\{\delta_k\}_{k>0}$ such that $\lim_k\frac{n_2(\delta_k)}{\delta_k}=\infty$. Then taking the limit of the expression for n_2 along δ_k, we arrive at the contradiction that $(1-z_2)(l-1)-z_2\geq 0$. So suppose now that there is a sequence $\{\delta_k\}_{k>0}$ such that $\lim_k\frac{n_2(\delta_k)}{\delta_k}=K$ for some number $K\in[0,\infty)$. Then, again taking the limit of the expression for n_2 along the sequence δ_k delivers that $Kz_2=l(1-z_2)+K(l-1)(1-z_2)$, and rearranging delivers the result.

Rearranging the upper bound for v_1 delivers that,

$$v_1\leq\frac{n_2e^{-(r_1+\rho)\tau}}{1-(1-n_2)e^{-(r_1+\rho)\tau}}.$$

Note that, $\lim_{\tau\to 0}\frac{1-e^{-(r_1+\rho)\tau}}{\tau}=r_1+\rho$. Combining the above inequality with the previous argument that $\lim_{\delta\to 0}\frac{n_2(\delta)}{\delta_k}=K$, delivers that, $\limsup_{\tau\to 0}v_1(\tau)\leq\frac{K\rho}{K\rho+\rho+r_1}$. Clearly then, there is a $\bar{\rho}>0$ and $\bar{\tau}>0$ such that if $\rho<\bar{\rho}$ and $\tau<\bar{\tau}$, then v_1 obtained from the strategy profile for which $p=1,q_1=1,q_2=0$ is less than $1-\theta_2$, which is a contradiction to the initial hypothesis that $v_1>1-\theta_2$.

Case 2: Suppose that $v_2 > 1-\theta_1$. Then similar to Case 1, it should be the case that $v_1 < 1-\theta_2$ and there is a unique equilibrium of $\Gamma(0,c,v)$ that induces trade probabilities $p=1, q_1=0, q_2=1$. In this case, the steady-state equation for n_1 becomes:

$$n_1 = \frac{\delta(1-z_1)}{z_1+\delta(1-z_1)}.$$

Hence, $\lim_{\delta\to 0}\frac{n_1(\delta)}{\delta} = \frac{1-z_1}{z_1}$. The rest of the argument replicates the argument for Case 1 to show that if ρ and τ are sufficiently small, then $v_2 < 1-\theta_1$, yielding a contradiction.

This establishes that if $\rho < \bar{\rho}$ and $\tau < \bar{\tau}$, then $v_1 \leq 1-\theta_2$ and $v_2 \leq 1-\theta_1$. Moreover, at least one of the inequalities have to be a strict inequality as we argued in the proof of Theorem 1. Because one of the inequalities is strict, in any SBU equilibrium, $p=1, q_i=1, q_j \in (0,1]$. Hence, all SBU equilibria have positive delays, no side has one-sided reputation building, and both inflexible types are traded with positive probability.

Now we will prove the second part of the Theorem, *i.e.* the case in which $l(1-z_2) > 1$. Consider the SBU equilibrium asserted in the theorem. In the bargaining stage, $T=0$, player 1 opts-out immediately, and the flexible type of player 2 yields immediately at $t=0$, and the resulting payoff of player 1 is the solution to the equation $v_1 = (n_2\theta_1 + (1-n_2)v_1)e^{-(r_1+\rho)\tau}$. Since if $v_1 > 1-\theta_2$, then the asserted behaviour is an equilibrium, it suffices to show that when the trade probabilities are $p=1, q_1=1, q_2=0$, then the resulting steady-state fraction of flexible firms from this configuration of trade probabilities is high enough and yields outside option value which satisfies $v_1 > 1-\theta_2$. Recall that the steady-state fraction calculated for this configuration of trade probabilities is:

$$n_2 = \frac{(1-z_2)(\delta l + (1-\delta)(l-1)n_2)}{\delta l + (1-\delta)(l-1)n_2 + z_2(1-\delta)}.$$

We now analyse the limit value of n_2 as $\tau \to 0$. Consider a sequence $\{\tau_k\}_{k>0}$ such that $\tau_k \to 0$ (equivalently, as $\delta_k \to 0$). We will first argue that there is no subsequence in which $\lim_k \frac{n_2(\tau_k)}{\delta_k} < \infty$. On the way to a contradiction, suppose that such a subsequence exists. Then, from the proof of Case 1 of this theorem, it is readily seen that if $\lim_k \frac{n_2(\tau_k)}{\delta_k} = K$, then $Kz_2 = l(1-z_2) + K(l-1)(1-z_2)$, which delivers a unique solution for K which is a negative number, and is a contradiction because $n_2(\delta) > 0$ for every $\delta > 0$. Since, along every subsequence, $\lim_k \frac{n_2(\tau_k)}{\delta_k} = \infty$, taking the limit of the expression for n_2 as $\delta \to 0$ yields, $\lim_{\delta\to 0} n_2(\delta_k) = 1 - z_2 - \frac{z_2}{l} > 0$. Therefore, there exists $\bar{\rho} > 0$ and $\bar{\tau} > 0$ such that $v_1 > 1-\theta_2$ as long as $\rho < \bar{\rho}$ and $\tau < \bar{\tau}$, completing the proof. ‖

B. THE DISCRETE-TIME ECONOMY

In this section, we first define the discrete-time bargaining stage game $\Gamma(\Delta, c, v)$. In lemmata B.1 through B.5 we characterize PBE for the stage-game $\Gamma(\Delta, c, v)$. In Lemma B.6 we show that there is no search equilibrium in which only the flexible types reach an agreement, as the search frictions disappear. In subsection B.2, we use our characterization to show that there is a sequence of equilibria of the discrete time economy that converges to the SBU equilibria of the continuous time model which we analysed in the main text.

B.1. *The discrete-time alternating offers bargaining game*

In the bargaining game, the worker is the proposer in the odd periods and the firm is the proposer in the even periods. The proposer can make an offer or can opt-out and terminate the bargaining relationship. If the proposer chooses to make an offer, then he/she proposes a division of the unit surplus. The responder can accept the offer, reject the offer or can opt-out and terminate the bargaining relationship. If agent i rejects agent j's offer, then agent i becomes the proposer after $\Delta > 0$ units of time. The bargaining game can terminate without an agreement because an agent voluntarily opts-out (*i.e.* due to an endogenous break-up). If the bargaining game terminates without an agreement, then both agents return to the unmatched population after $\tau > 0$ units of time. The extensive form of the bargaining game is given in Figure A1.

Let h^t denote a period t history for agent i which contains all the information that agent i has observed up to period t and let H denote the set of all histories for player i. A strategy for a flexible player i is a function $\sigma_i : H \to [0,1] \cup \{accept, reject, opt-out\}$. The strategy for player i, $\sigma_i(h^t) \in [0,1] \cup \{opt-out\}$, if player i is making an offer in any period t history h^t and $\sigma_i(h^t) \in \{accept, reject, opt-out\}$, if player i is responding in any period t history h^t. A behaviour strategy is similarly defined but has the player randomizing over the action choices. We focus on equilibria in which agents of the same type and belonging to the same side of the market use the same strategy, *i.e.* we focus on symmetric equilibria. A belief for agent i is a function $\mu_j : H \to [0,1]$ that gives the probability that agent i places on his bargaining partner j being the inflexible type, when player i is bargaining with agent j.[21]

21. At histories where player i is not bargaining with another agent, we set the belief function to equal the steady-state frequency of inflexible type j, *i.e.* $\mu_j = c_j$.

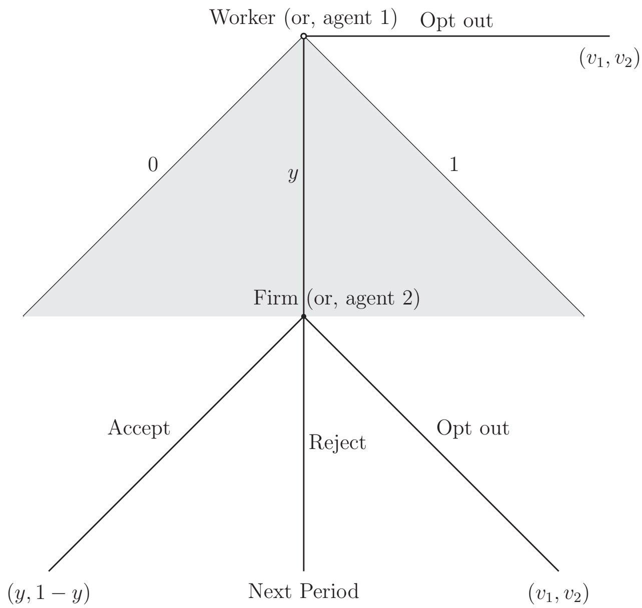

FIGURE A1

This depicts the bargaining game in any odd period where the worker speaks first. The option value for agent i of opting out and returning to the unmatched matched population is equal to v_i

Lemma B.1 considers a situation where both outside options are dominated by the inflexible type demands. In this case, both players would rather trade with an inflexible type than take the outside option. This game is identical to the bargaining game analysed by Abreu and Gul (2000). Abreu and Gul (2000) showed that all PBE of this game converge to a war of attrition.

Lemma B.1 (Two-sided reputation). *Suppose that $c_1 > 0$, $c_2 > 0$. If both player 1 and player 2's outside options are **compatible** with $1-\theta_2$ and $1-\theta_1$ respectively, i.e. $v_1 < 1-\theta_2$ and $v_2 < 1-\theta_1$, then $\lim_{\Delta\to 0} U_i(\Delta, \sigma(\Delta)) = (1-b_j)\theta_i + b_j(1-\theta_j)$ for any PBE $\sigma(\Delta)$ of $\Gamma(\Delta, c, v)$ where the definition of b_i is given in equation* (11).

Proof See Abreu and Gul (2000) Proposition 4. ‖

Lemma B.2 considers a situation where each player's outside option exceeds the payoff from trading with a inflexible type. Under this scenario, the incentive to mimic the inflexible type is eliminated for both players since their opponent never accepts the demand of an inflexible type. However, once both players reveal rationality, the unique PBE of the bargaining game results in the Rubinstein (see Rubinstein (1982)) outcome. This result, established in Compte and Jehiel (2002), is summarized in the following lemma. We define the Rubinstein payoffs as $u_1^*(\Delta) \equiv \frac{1-e^{-r_2\Delta}}{1-e^{-(r_1+r_2)\Delta}}$ and $u_2^*(\Delta) \equiv \frac{e^{-r_2\Delta}(1-e^{-r_1\Delta})}{1-e^{-(r_1+r_2)\Delta}}$.

Lemma B.2. *Suppose that $c_1 > 0$, $c_2 > 0$, $v_1 < u_1^*(\Delta)$ and $v_2 < u_2^*(\Delta)$. If both player 1 and player 2's outside options are **incompatible** with $1-\theta_2$ and $1-\theta_1$ respectively, i.e. $v_1 > 1-\theta_2$ and $v_2 > 1-\theta_1$, then the players receive their Rubinstein payoffs, i.e. $U_1(\sigma|n) = u_1^*(\Delta)$ and $U_2(\sigma|n) = u_2^*(\Delta)$, in any PBE σ of $\Gamma(\Delta, c, v)$*

Proof See Compte and Jehiel (2002) Proposition 5. ‖

Below we summarize findings by Compte and Jehiel (2002) for the bargaining stage-game $\Gamma(\Delta, c, v)$, where $c_1 > 0$, $c_2 = 0$ and $v_i < 1-\theta_j$. We assume that player 1 is potentially an inflexible type, player 2 is known to be flexible with probability one, and both players' outside options are worse than yielding to the inflexible types. Our development follows Compte and Jehiel (2002) Appendix A where all the stated results can be found.

Lemma B.3. *Suppose that* $c_1 > 0$, $c_2 = 0$, $v_1 < 1-\theta_2$ *and* $v_2 < 1-\theta_1$. *Then,* $\lim_{\Delta\to 0} U_1(\Delta,\sigma(\Delta)) = \theta_1$ *and* $U_2(\Delta,\sigma(\Delta)) = 1-\theta_1$ *in any PBE* $\sigma(\Delta)$ *of* $\Gamma(\Delta,c,v)$.

Proof See the development in Myerson (1991), Chapter 8, Theorem 8.4; or Abreu and Gul (2000) Lemma 1; or Compte and Jehiel (2002), Proposition 2. ‖

Let $\gamma_i = e^{-r_i\Delta}$. Following Compte and Jehiel (2002), Appendix A, define $v^n = [\gamma_1]^{2n+1}\theta_1$, $\rho = \frac{1-[\gamma_1]^2}{1-[\gamma_2]^2}$, $\pi^0 = 1-\mu^0$, $w^0(\mu) = \max\{(1-\mu)(1-v^0) + \mu\gamma_2(1-\theta_1), 1-\theta_1\}$ and

$$\mu^0 : \gamma_2 w^0(\mu^0) = 1-\theta_1 \text{ if } \gamma_2 w^0(0) \geq 1-\theta_1, \text{ and } \mu^0 = 0 \text{ otherwise.}$$

Let N be the largest integer for which $[\gamma_1]^{2N}\theta_1 > u_2^*(\Delta)$. Consider the sequence $\{\pi^n, \mu^n, w^n, w^n(.)\}_{0\leq n\leq N}$ defined recursively by $\pi^{n+1} = \frac{w^n}{w^n+\rho v^n}$, $\mu^{n+1} = \Pi_{k\leq n+1}(1-\pi^k)$, $w^{n+1} = \pi^{n+1}(1+(\rho-1)v^n)$, and $w^{n+1}(\mu) = (1-\mu/\mu^n)(1-v^{n+1}) + (\mu/\mu^n)[\gamma_2]^2 w^n$. In this sequence, $w^n = w^n(\mu^n) = w^{n-1}(\mu^n)$. The following lemma shows that player 2's equilibrium payoff is a continuous and non-decreasing function of μ, where μ is the probability that player 1 is an inflexible type, *i.e.* μ is player 1's reputation level. The strength of this lemma is that it shows player 2's equilibrium payoff is independent of which equilibrium is played and which history has been reached in the game. It is completely determined by player 1's reputation level and by whether player 2 is a proposer or a responder.

Lemma B.4. *Suppose that* $\mu > \mu^N$. *Let* $u_2(\mu)$ *be the function that coincides with* $w^n(\mu)$ *on each interval* $(\mu^{n+1}, \mu^n]$, $n \in \{0, .., N-1\}$.

(*i*) *Player 1 proposes* θ_1 *in all odd periods,*
(*ii*) *In any even period, if player 1's reputation level* $\mu \in (\mu_{n+1}, \mu_n)$, *then player 2 proposes* v_n,
(*iii*) *In any even period if player 1 is the inflexible type with probability* μ, *then player 2's equilibrium payoff is equal to* $u_2(\mu)$,
(*iv*) *In any odd period where player 1 has proposed* θ_1, *if player 1 is the inflexible type with probability* μ, *then player 2's equilibrium payoff is equal to* $\max\{1-\theta_1, e^{-r_2\Delta}u_2(\mu)\}$

in any PBE σ *of* $\Gamma(\Delta,c,v)$.

Proof See Compte and Jehiel (2002), Proposition 10. ‖

Note that $\mu^N \to 0$ as $\Delta \to 0$. The previous lemma also pins down player 1's payoff at all reputation levels possible in the game except the cut-off reputation level μ^n. This is because player 2 always offers v^n when $\mu \in (\mu^{n+1}, \mu^n)$ and player 1 always randomizes between accepting and rejecting. Consequently, in any even period where player 1's reputation is $\mu \in (\mu^{n+1}, \mu^n)$, player 1's equilibrium payoff is v^n. When $\mu = \mu^n$, player 2 may offer v^n or v^{n-1}. Compte and Jehiel (2002) construct PBE where player 2 randomizes between v^n and v^{n-1} if $\mu = \mu^n$. If $\mu = \mu^n$, then for any $q \in [0,1]$ there is a PBE where player 2 offers v^n with probability q and offers v^{n-1} with probability $1-q$. Therefore, if $\mu = \mu^n$, then player 1's equilibrium payoff set is the convex and closed interval $[v^n, v^{n-1}]$.

Lemma B.5. *There exists a PBE for the game* $\Gamma(\Delta,c,v)$. *The equilibrium payoff set for player 1, viewed as a (possibly multi-valued) function of* μ *is an upper-hemi-continuous compact and convex valued correspondence.*

Proof See the above discussion and Compte and Jehiel (2002), Proposition 11. ‖

We now proceed to show that, in the market where the bargaining-stage is a discrete-time bargaining model, there is no search equilibrium in which only the flexible types reach an agreement, as the search frictions disappear.

Lemma B.6. *Fix* $z_1, z_2 > 0$, *and consider a search equilibrium s of the market* $E(\Delta,\tau,\rho,l,z)$. *There exist* $\bar{\Delta} > 0$, $\bar{\rho} > 0$ *and* $\bar{\tau} > 0$ *such that if* $\Delta < \bar{\Delta}$, $\rho < \bar{\rho}$ *and* $\tau < \bar{\tau}$, *then either* $v_1 \leq 1-\theta_2$ *or* $v_2 \leq 1-\theta_1$. *Moreover, either* $q_1 > 0$ *or* $q_2 > 0$.

Proof On the way to a contradiction, suppose that $v_1 > 1-\theta_2$ and $v_2 > 1-\theta_1$. There exist $\underline{c}_1, \underline{c}_2 > 0$ such that in any search equilibrium, $c_1 \geq \underline{c}_1$ and $c_2 \geq \underline{c}_2$. Therefore, it follows from Compte and Jehiel (2002) that there exists a $\bar{\Delta} > 0$ such that, if $\Delta < \bar{\Delta}$, then there is a unique Perfect Bayesian equilibrium of $\Gamma(\Delta,c,v)$, and in this equilibrium, only the flexible types trade, agreement is achieved in the first period, and the inflexible types are not traded with. In particular, $p = 1, q_1 = 0, q_2 = 0$.

In the following, we proceed similar to the proof of Theorem 3 to show that, as $\delta \to 0$, then the inflexible types become sufficiently likely that the search for a flexible opponent becomes very high for at least one side. Consider a sequence $\{\delta_k\}_{k>0}$ such that $\lim_k \delta_k = 0$. There are two cases to consider.

Case 1: There is a subsequence of $\{\delta_k\}_k$ in which $\frac{n_1(\delta_k)n_2(\delta_k)}{\delta_k} \to \infty$. In this case, rearranging the steady-state equation for $n_1(\delta_k)$ by plugging $p=1, q_1=q_2=0$, we obtain $n_1(\delta_k)\delta_k + n_1(\delta_k)z_1(1-\delta_k)n_2(\delta_k) = (1-z_1)\delta_k$. However, if $\frac{n_1(\delta_k)n_2(\delta_k)}{\delta_k} \to \infty$, then this equation has no solution as $k \to \infty$, which is a contradiction.

Case 2: There is a subsequence of $\{\delta_k\}_k$ in which $\frac{n_1(\delta_k)n_2(\delta_k)}{\delta_k} \to K < \infty$. Then, taking the limit of the above equation and canceling out δ_k from both sides yields the equality: $\lim_k n_1(\delta_k) = 1 - z_1 - z_1 K$. Since $\lim_k n_1(\delta_k) \geq 0$, $K \leq \frac{1-z_1}{z_1}$. Therefore, $\lim_k \frac{n_2}{\delta_k} = \frac{K}{1-z_1-z_1 K}$. There are now two further cases to consider, either $\limsup_k \frac{n_1(\delta_k)}{\delta_k} < \infty$ or $\limsup_k \frac{n_1(\delta_k)}{\delta_k} = \infty$. In the latter case, pick the subsequence on which the division is divergent. Then, taking the limit of the expression for n_2 delivers that $\limsup_k \frac{n_2(\delta_k)}{\delta_k} < \infty$. Now, applying the equilibrium value inequality that $v_i \leq e^{-\tau(r_i+\rho)}(n_j + (1-n_j)v_i)$, and using the same limit calculation in the proof of Theorem 3 yields in $v_1 \leq 1-\theta_2$ or $v_2 \leq 1-\theta_1$ if $\rho < \bar{\rho}$ and $\tau < \bar{\tau}$, for some $\bar{\rho} > 0$ and $\bar{\tau} > 0$. ‖

B.2. *SBU equilibria: existence and convergence*

First, suppose that an SBU equilibrium s of $E(0,\tau,\rho,l,z)$ is a one-sided reputation equilibrium, *i.e.* if $\Lambda_i(0) > 1-\theta_j$, where $\Lambda_i(q_j)$ is the function defined in equations (A.10) and (A.17). Then in the discrete-time bargaining games, if Δ is sufficiently small (*i.e.* if $\Delta < \bar{\Delta}$ for some $\bar{\Delta} > 0$), then there is a one-sided reputation equilibrium where $v_i > 1-\theta_j$, flexible type of player j yields to player i right away, and player i opts-out against his opponent if player j has not accepted his inflexible demand. This is because, the steady-state equations do not depend on Δ, and given the proposed strategies the distribution of types in equilibrium s is the same as that in the discrete-time model with one-sided reputation equilibrium strategies. Therefore, the outside options in the two configurations coincide. Moreover, because $v_i > 1-\theta_j$ and $v_j < 1-\theta_i$ in this candidate equilibrium, it follows from Compte and Jehiel (2002) that the one-sided reputation strategies constitute an equilibrium of the bargaining game for every Δ sufficiently small. Since such an equilibrium exists for every $\Delta < \bar{\Delta}$, the equilibrium outcomes of such equilibria converge to the one-sided reputation SBU equilibrium.

Similarly, if $\Lambda_i(1) < 1-\theta_j$ for both players, and if Δ is sufficiently small (*i.e.* if $\Delta < \bar{\Delta}$ for some $\bar{\Delta} > 0$), then the Abreu and Gul (2000) bargaining outcome in the discrete-time game delivers outside options that are strictly less than yielding to an inflexible opponent, and hence these are sustained as limits of equilibrium outcomes of the discrete-time bargaining games.

In this section, we will prove the more difficult case. We will show that the unique SBU equilibrium that we constructed in the main text for $l=1$, $z_i < z^*$ and $\tau = 0$ can be approximated as equilibrium outcomes of discrete-time bargaining games as when Δ and τ are positive but sufficiently small. In particular, we will be showing the existence of equilibria of the market $E(\Delta,\tau(\Delta),\rho,1,z)$ whose outcomes coincide with the equilibrium outcomes of $E(0,0,\rho,1,z)$ as $\Delta \to 0$ and $\tau(\Delta) \to 0$. Let $\tau(\Delta)$ have the property that $\tau(\Delta) > \Delta^{1/3}$ and $\lim_{\Delta\to 0}\tau(\Delta) = 0$.

B.2.1. ς-SBU equilibria in discrete bargaining games. For any given strategy profile σ, let $f_1(t)$ denote the probability that player 1 reveals rationality in period t of the bargaining stage. A ς-SBU equilibrium is a search equilibrium with the following properties

(i) Player 2 trades with inflexible type 1 with probability 1,
(ii) Player 1 opts-out with positive probability in a period t only if $\sum_{s>t} f_1(s) < \varsigma$
(iii) $v_1 = 1-\theta_2$ and $v_2 < 1-\theta_1$.

B.2.2. Existence. Below we show the existence of ς-SBU equilibria.

Theorem B.1. *Let s be the SBU equilibrium of $E(0,0,\rho,1,z)$ as described in subsection 3.1. Then there is a $\bar{\Delta} > 0$ such that for any $\Delta < \bar{\Delta}$, and for any $\varsigma > 0$, a ς-SBU equilibrium of the market $E(\Delta,\tau(\Delta),\rho,1,z)$ exists.*

Proof We define an "*alternative*" bargaining game in discrete time, prove that a search equilibrium exists if the players play this *alternative* game in the bargaining stage, and show that this equilibrium is also an equilibrium for the original search economy.

Step 1. The alternative game.

Given exogenous payoff function $w : \mathbb{N} \to \mathbb{R}^2$, exogenous total break-up probability a, vector of outside options v, and vector of inflexible type probabilities c we define the *alternative* game $\hat{\Gamma}(a,c,v,w)$. In the *alternative* game player 1 moves first in the odd periods and player 2 moves first in the even periods. The player that moves first has two actions available, $\{R(eveal), I(nsist)\}$. If the player that moves first chooses R, then the game ends and payoffs are realized. If the player that moves first chooses I, then the follower picks action from $\{R(eveal), I(nsist)\}$. If she choses R, then the game ends and otherwise the game progresses to the next period. Also, at any node in period t where player 1 moves the game ends with probability $\alpha_1(t)$. The opt-out probability $\alpha_1(t)$ is a function of a and strategies. We define $\alpha_1(t)$ in Step 3. The function $w:\mathbb{N}\to\mathbb{R}^2$ determines payoffs (before discounting) to each player from revealing and being revealed to, at any period t, after a play of R by the player who speaks first. In a period where j moves first, a typical element of w, denoted $w(t)=(w_i^j,w_j^j)(t)$ where $w_i^j(t)$ is the payoff to i from j revealing in period t, and similarly $w_j^j(t)$ is the payoff to player j of revealing in period t. If the player that speaks second, player j, reveals, then player i receives payoff θ_i and player j receives payoff $1-\theta_i$. Let $\gamma_i^t:=e^{-rt\Delta}$. Then, if there is a break-up in a period t, then the agents receive v_i as their (undiscounted) payoff, and $\gamma_i^t v_i$ as their discounted payoff. In this game, the inflexible types never opt-out or take action R, and player i is the inflexible type with probability c_i.

The alternative game is interpreted as follows: the strategy insist corresponds to player i asking for θ_i and rejecting an offer of θ_j by player j in the original game. Reveal corresponds to player i proposing something different than θ_1 but on an equilibrium path for the game with one-sided incomplete information in the original game. The exogenous continuation payoffs w are chosen from the set of equilibrium payoff vectors for the game with one-sided incomplete information. The exogenously given opt-out probability a is incorporated into the game so that player 1 only opts-out against the inflexible type.

Step 2. Strategies in the alternative game. Let $\mathcal{F}=\{F:\mathbb{N}\to[0,1], F \text{ non-decreasing}\}$, that is $\mathcal{F}$ is the set of all sub-probability distribution functions over the set of natural numbers. Let $F(\infty)=\lim_{t\to\infty}F(t)$. Let f denote the density of F, *i.e.* $f(t)=F(t)-F(t-1)$. A strategy for player 1 is a function F_1 such that $F_1\in\mathcal{F}$, and $\sum_t f_1(t)\le 1-c_1-a$. A strategy for player 2 is a function $F_2\in\mathcal{F}$ such that $\sum_t f_2(t)\le 1-c_2$.

Step 3. For any $F\in\mathcal{F}$ for player 1 let t_ς denote the first period such that $F(t)\ge F(\infty)-\varsigma$. For any exogenously given total opt-out probability $a\in[0,1]$ let

$$\alpha_1(t,F,a)=\begin{cases} a\frac{F(t_\varsigma)-(F(\infty)-\varsigma)}{\varsigma} & \text{for } t=t_\varsigma,\\ a\frac{f(t)}{\varsigma} & \text{for } t>t_\varsigma,\\ 0 & \text{for } t<t_\varsigma.\end{cases}$$

Step 4. Utilities in the alternative game. Suppose player i uses strategy F_i. Define $\alpha_2(.)=0$. In the following, we drop the dependence of α_1 on F_1 and a when this does not cause any ambiguity. In this game, the payoff to player i from revealing at time t where player i is the player to propose

$$U_i(F,a,c,v,w,t)=\sum_{s<t}\gamma_i^s(f_j(s)w_i^j(s)+\alpha_j(s)v_i)+(1-F_j(t-1)-\sum_{s<t}\alpha_j(s))\gamma_i^t w_i^i(t).$$

The payoff to player i from revealing at time t where player i is the player to respond

$$U_i(F,a,c,v,w,t)=\sum_{s\le t}\gamma_i^s(f_j(s)w_i^j(s)+\alpha_j(s)v_i)+(1-F_j(t)+\sum_{s\le t}\alpha_j(s))\gamma_i^t(1-\theta_j).$$

Step 5. The fixed point operator Φ. Consider the SBU equilibrium s, where player 1 trades with the inflexible opponent with probability $q_2^*\in(0,1)$. Let p^* be the constant defined in equation (A.3). Now, define the correspondence Φ such that $(F',a',c',v',w')\in\Phi(F,a,c,v,w)$ if and only if

$$\frac{a'}{1-c_1}=\begin{cases}1-p^* & \text{if } v_1<1-\theta_2,\\ 0 & \text{if } v_1>1-\theta_2,\\ [0,1-p^*] & \text{otherwise.}\end{cases}$$

$$F_1'\in\arg\max_{\{\hat{F}_1\in\mathcal{F}:\sum_t\hat{f}_1(t)\le 1-c_1-a\}}\sum_{t\ge 0}\gamma_1^t\hat{f}_1(t)U_1(F,a,c,v,w,t)$$

$$F_2'\in\arg\max_{\{\hat{F}_2\in\mathcal{F}:\sum_t\hat{f}_2(t)\le 1-c_2\}}\sum_{t\ge 0}\gamma_2^t\hat{f}_2(t)U_2(F,a,c,v,w,t)$$

$$v_1'=e^{-\tau(r_1+\rho)}\sum_{t\ge 0}\gamma_1^t\left(\frac{f_1(t)}{1-c_1}U_1(F,a,c,v,w,t)+\frac{\alpha_1(t,F_1,a)v_1}{1-c_1}\right)$$

$$v_2'=e^{-\tau(r_2+\rho)}\min\{1-\theta_1,\sum_{t\ge 0}\gamma_2^t\frac{f_2(t)}{1-c_2}U_2(F,a,c,v,w,t)\}$$

and (c_1',c_2') are the unique solutions to the following equations:

$$\frac{n_1'}{c_1'}=\frac{1-z_1}{z_1}\frac{\delta+(1-\delta)n_2'q_1'}{\delta+(1-\delta)(n_2'p'+c_2'q_2')},\tag{B.1}$$

$$\frac{n_2'}{c_2'}=\frac{1-z_2}{z_2}\frac{\delta+(1-\delta)n_1'q_2'}{\delta+(1-\delta)(n_1'p'+c_1'q_1')}.\tag{B.2}$$

where (p', q_1', q_2') are the probability that two flexible types trade, flexible type of 2 trades with the inflexible type of player 1, and flexible type of player 1 trades with inflexible type player 2, respectively, given the yield strategy F and opt-out strategy α_1.

Also, let $\mu_i(F,a,t)$ denote the probability that player i is an inflexible type given that player i has not revealed rationality in history h^t. The posterior probability μ_i is obtained using Bayes' rule conditioning on strategies $(F_1, F_2, \alpha_1(F_1,a))$. Notice $\mu_i(F,a,t)$ is a continuous function of (F,a). Let

$$(w_i^j, w_j^j)(t)' = \left\{U_i(\mu_i(F,a,t)), U_j(\mu_i(F,a,t))\right\},$$

where $\left\{U_i(\mu_i(F,a,t)), U_j(\mu_i(F,a,t))\right\}$ denotes the set of perfect equilibrium payoff vectors in the bargaining game with one-sided incomplete where player i reputation level is $\mu_i(F,a,t) > 0$. Recall that $(U_1(\mu_1), U_2(\mu_1))$ is an upper-hemi continuous, convex and compact valued correspondence (as a function of μ_1) by Lemma B.5.

The correspondence Φ, as defined above is clearly upper hemi-continuous, compact and convex-valued (in the product topology). Consequently, Glicksberg's fixed point theorem implies that a fixed point, (F,a,c,v,w) exists.

Step 6. The remaining steps show that if ς and Δ are sufficiently small, then $(F, \alpha_1(F_1,a), c, v)$ is a search equilibrium in the discrete economy, v is the vector of values in this equilibrium, and the continuation equilibrium once one player has revealed is chosen from the set of equilibria of the game with one-sided incomplete information such that payoffs are according to w.

Step 7. There exist numbers $\underline{c_1}, \underline{c_2} > 0$ such that any vector c that is part of some fixed point of the operator Φ satisfies $c_i > \underline{c_i}$ for $i \in \{1,2\}$. Therefore, there is a $\kappa > 0$ such that $|w_i^j - \theta_i|, |w_j^j - (1-\theta_i)| < \kappa\Delta$.

Any fixed point c has to satisfy the steady-state equations, and these equations deliver a strictly positive lower bound for c_1, c_2 across all trade probabilities p, q_1, q_2.

Step 8. The probability that flexible player 1 and flexible player 2 trade is at least $1 - C\Delta$, where C is a constant independent of Δ.

Player 2 will complete her yielding by the latest in period $t_\varsigma + 1$ for sufficiently small ς. This is because the probability that player 1 yields in some period after t_ς is bounded above by ς. Consequently, player 2 will do strictly better by completing yielding in period $t_\varsigma + 1$ than in any period $t > t_\varsigma + 1$.

Suppose that player 2 reveals rationality with a total probability $p > C\Delta$ in periods t_ς and $t_\varsigma + 1$. Observe that player 1 reveals with positive probability in period t_ς by the definition of this period.

Suppose that period t_ς is a period where player 1 is proposing. Instead of revealing rationality in t_ς, player 1 can wait until $t_\varsigma + 1$, and reveal rationality with certainty then, if player 2 has not revealed in his proposal in period $t_\varsigma + 1$. If this strategy does any better than revealing rationality in t_ς, then we arrive at a contradiction to the observation that player 1 reveals rationality in period t_ς. This implies that

$$1 - \theta_2 + \Delta\kappa \geq e^{-r_1\Delta}(1-p)(1-\theta_2) + \theta_1 p.$$

This inequality cannot hold for C sufficiently large.

Suppose that period t_ς is a period where player 1 is responding. If player 2 reveals with probability p in period $t_\varsigma + 1$, then player 1 would be better of not revealing in period t_ς, which is a contradiction. Since player 1 is responding in period t_ς, the probability with which the flexible types trade is at least $1 - C\Delta$ for a sufficiently large C, which is independent of Δ.

Step 9. The bounds in the operator Φ are not binding, $\frac{a}{1-c_1} \in (0, 1-p^*)$, $v_1 = 1-\theta_2$, and $v_2 < 1-\theta_1$. Consequently, $(F, \alpha_1(F_1,a), c, v)$ is a search equilibrium of the market $E(\Delta, \tau, \rho, l, z)$, and v is the vector of values in this equilibrium, of the economy where the bargaining stage game is the original bargaining game and the continuation equilibrium once one player has revealed is chosen from the set of equilibria of the game with one-sided incomplete information such that payoffs are according to w.

Suppose that $v_1 < 1-\theta_2$, then $\frac{a}{1-c_1} = 1-p^*$. Revelations at any period need to occur at rates that converge to λ_1 and λ_2 as $\Delta \to 0$, by Lemma B.1. However, the opt-out probability for flexible player 1, $1-p^*$ induces the trade probability between the flexible player 1 and inflexible player 2 to be p^*. But because $\Lambda_1(p^*) > 1-\theta_2$, when Δ is sufficiently small, this would imply that $v_1 > 1-\theta_2$ which is a contradiction to the initial hypothesis that $v_1 < 1-\theta_2$. A similar argument rules out $v_1 > 1-\theta_2$. Consequently, $v_1 = 1-\theta_2$ and $\frac{a}{1-c_1} \in (0, 1-p^*)$. Notice that since $\tau(\Delta) > \Delta^{1/3}$, and since player 2 is the first player that reveals rationality with a positive probability, $v_2 < 1-\theta_2$. ‖

B.2.3. Convergence of discrete-time ς-SBU equilibria to SBU equilibria. Let $\bar{\Delta} > 0$ be the cutoff chosen in Theorem B.1. Let $(F_1^n, \alpha^n, F_2^n, v^n, c^n)$ denote a sequence of ς^n-SBU equilibria for the economy where the period length is $\Delta^n > 0$ where $\Delta^n < \bar{\Delta}$. Suppose $\lim_{n\to\infty} \Delta^n = 0$, also, suppose that $\lim_{n\to\infty} \varsigma^n = 0$. Such a sequence of equilibria exists by Theorem B.1.

Remark B.1. *By step 7, $c_i^n \geq \underline{c_i}$. Following a similar argument to Abreu and Gul (2000), Lemma 1 (also see Step 1, on page 111), we obtain that there exists a time T such that $F^n(T) + \sum_{t \leq T} \alpha^n(t) = 1 - c_i^n$ and $F_2^n(T) = 1 - c_2^n$, for all n. Hence,* *the sub-probability distributions* (F_1^n, α^n, F_2^n) *have uniformly bounded support* $[0,T]$*. Consequently, Helly's theorem (Billingsley (1995), Theorem 25.9) implies that* $(F_1^n, \alpha^n, F_2^n, v^n, c^n)$ *has a convergent subsequence. Let* (F_1, α, F_2, v, c) *denote a sub-sequential limit.*

Theorem B.2. *The limit* (F_1, α, F_2, v, c) *is the unique SBU equilibrium for the continuous time bargaining stage-game.*

Proof Step 1. F_1 and F_2 do not have common discontinuity points. Also, the function $G = F_1 + \alpha$ and the function F_2 do not have common discontinuity points.

Step 2. Let $U_1^n = \int\int U_1(t,k) dG^n(t) dF_2^n(k)$ and $U_2^n = \int\int U_2(t,k) dF_1^n(t) dF_2^n(k)$ where

$$U_i(t,k) = \begin{cases} \theta_i & \text{if } t > k, \\ 1-\theta_j & \text{if } t < k, \\ 1/2 & \text{if } t = k. \end{cases}$$

F_1, F_2, and G do not have common discontinuity points. Consequently, Billingsley (1995), Theorem 29.2 and Exercise 29.2 imply that $\lim U_1^n = \int\int U_1(t,k) dG(t) dF_2(k)$ and $\lim U_2^n = \int\int U_2(t,k) dF_1(t) dF_2(k)$. Also, $v_1 = \frac{\lim U_1^n}{1-c_1}$ and $v_2 = \frac{\lim U_2^n}{1-c_2}$.

Step 3. The functions (F_1, α, F_2) together with (c, v) comprise the unique SBU equilibrium for the continuous time war of attrition.

The vector c^n and a^n satisfy the steady-state equations for all n. Hence, c and a satisfy the steady-state equations.The probability that flexible player 1 and flexible player 2 successfully agree, $p^n \geq 1 - C\Delta^n$ where C is independent of Δ^n. Hence, $\lim_n p^n = 1$. The outside option values satisfy, $v_1^n = 1-\theta_2$ and $v_2^n \leq (1-\theta_1)e^{-(r_2+\rho)\tau(\Delta)}$ for all n. Because $\lim \tau(\Delta) = 0$, we have $v_1 = 1-\theta_2$ and $v_2 = 1-\theta_1$.

We now show that (F_1, α) and F_2 are mutual best responses. F_1 does not jump at T. In the continuous time war of attrition, if player 1 is behaving according to (F_1, α), then for each ϵ, there is a N such that for all $n > N$, F_2^n is an ϵ best response to (F_1, α) and consequently, since ϵ is arbitrary F_2 is a best response to (F_1, α). Also, the symmetric argument is true for player 2 showing that (F_1, α) is a best response to F_2. Proving that (F_1, α) and F_2 is an equilibrium. Since the war of attrition has a unique equilibrium, (F_1, α) and F_2 coincides with this equilibrium. This argument is identical to Abreu and Gul (2000), proof of Proposition 4, on page 114 where a more detailed proof may be found. ||

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